FACTORIZABLE ENRICHED CATEGORIES AND APPLICATIONS

AURA BÂRDEŞ AND DRAGOŞ ŞTEFAN

Abstract. We define the twisted tensor product of two enriched categories, which generalizes various sorts of 'products' of algebraic structures, including the bicrossed product of groups, the twisted tensor product of (co)algebras and the double cross product of bialgebras. The key ingredient in the definition is the notion of simple twisting systems between two enriched categories. To give examples of simple twisted tensor products we introduce matched pairs of enriched categories. Several other examples related to ordinary categories, posets and groupoids are also discussed.

Contents

Introduction

The most convenient way to explain what we mean by the factorization problem of an algebraic structure is to consider a concrete example. Chronologically speaking, the first problem of this type was studied for groups, see for instance [Mai, Ore, Za, Sz, Tak]. Let G be a group. Let H and K denote two subgroups of G. One says that G factorizes through H and K if $G = HK$ and $H \cap K = 1$. Therefore, the factorization problem for groups means to find necessary and sufficient conditions which ensure that G factorizes through the given subgroups H and K. Note that, if G factorizes through H and K then the multiplication induces a canonical bijective map $\varphi : H \times K \to G$, which can be used to transport the group structure of G on the Cartesian product of H and K. We shall call the resulting group structure the bicrossed product of H and K, and we shall denote it by $H \bowtie K$. The identity element of $H \bowtie K$ is $(1, 1)$, and its group law is uniquely determined by the 'twisting' map

$$R : K \times H \to H \times K, \quad R(k, h) := \varphi^{-1}(kh).$$

Obviously, R is induced by a couple of functions $\triangleright : K \times H \to H$ and $\triangleleft : K \times H \to K$ such that $R(k, h) = (k \triangleright h, k \triangleleft h)$. Using this notation the multiplication on $H \bowtie K$ can be written as

$$(h, k) \cdot (h', k') = (h(k \triangleright h'), (k \triangleleft h')k') \,.$$

The group axioms easily imply that $(H, K, \triangleright, \triangleleft)$ is a matched pair of groups, in the sense of [Tak]. Conversely, any bicrossed product $H \bowtie K$ factorizes through H and K. In conclusion, a group G factorizes through H and K if and only if it is isomorphic to the bicrossed product $H \bowtie K$ associated to a certain matched pair $(H, K, \triangleright, \triangleleft)$.

2000 *Mathematics Subject Classification.* Primary 18D20; Secondary 18D10; 16Sxx.
Key words and phrases. Enriched category, twisting system, twisted tensor product, matched pair, bicrossed product.

Similar 'products' are known in the literature for many other algebraic structures. In [Be], for a distributive law $\lambda : G \circ F \to F \circ G$ between two monads, Jon Beck defined a monad structure on $F \circ G$, which can be regarded as a sort of bicrossed product of F and G with respect to the twisting natural transformation λ.

The twisted tensor product of two $\mathbb{K}$-algebras A and B with respect to a $\mathbb{K}$-linear twisting map $R : B \otimes_{\mathbb{K}} A \to A \otimes_{\mathbb{K}} B$ was investigated for instance in [Ma1], [Tam], [CSV], [CIMZ], [LPoV] and [JLPvO]. It is the analogous in the category of associative and unital algebras of the bicrossed product of groups. The classical tensor product of two algebras, the graded tensor product of two graded algebras, skew algebras, smash products, Ore extensions, generalized quaternion algebras, quantum affine spaces and quantum tori are all examples of twisted tensor products.

Another class of examples, including the Drinfeld double and the double crossed product of a matched pair of bialgebras, comes from the theory of Hopf algebras, see [Ma2]. Some of these constructions have been generalized for bialgebras in monoidal categories [BD] and bimonads [BV].

Enriched categories have been playing an increasingly important role not only in Algebra, but also in Algebraic Topology and Mathematical Physics, for instance. They generalize usual categories, linear categories, Hopf module categories and Hopf comodule categories. Monoids, algebras, coalgebras and bialgebras may be regarded as enriched categories with one object.

Our aim in this paper is to 'categorify' the factorization problem, i.e. to answer the question when an enriched category factorizes through a couple of enriched subcategories. Finding a solution at this level of generality would allow us to approach in an unifying way all factorization problems that we have already mentioned. Moreover, it would also provide a general method for producing new non-trivial examples of enriched categories.

In order to define factorizable enriched categories, we need some notation. Let $\boldsymbol{C}$ be a small enriched category over a monoidal category $(\boldsymbol{M}, \otimes, \mathbf{1})$. Let S denote the set of objects in $\boldsymbol{C}$. For the hom-objects in $\boldsymbol{C}$ we use the notation ${}_xC_y$. The composition of morphisms and the identity morphisms in $\boldsymbol{C}$ are defined by the maps ${}_xc_z^y : {}_xC_y \otimes {}_yC_z \to {}_xC_z$ and $1_x : \mathbf{1} \to {}_xC_x$, respectively. For details, the reader is referred to the next section. We assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are two enriched subcategories of $\boldsymbol{C}$. The inclusion functor $\boldsymbol{\alpha} : \boldsymbol{A} \to \boldsymbol{C}$ is given by a family $\{{}_x\alpha_y\}_{x,y\in S}$ of morphisms in $\boldsymbol{M}$, where ${}_x\alpha_y : {}_xA_y \to {}_xC_y$. If $\boldsymbol{\beta}$ is the inclusion of $\boldsymbol{B}$ in $\boldsymbol{C}$, then for x, y and u in S we define

$$ {}_x\varphi_y^u : {}_xA_u \otimes {}_uB_y \to {}_xC_y, \quad {}_x\varphi_z^y = {}_xc_y^u \circ ({}_x\alpha_u \otimes {}_u\beta_y). $$

Assuming that all S-indexed families of objects in $\boldsymbol{M}$ have a coproduct it follows that the maps $\{{}_x\varphi_y^u\}_{u\in S}$ yield a unique morphism

$$ {}_x\varphi_y : \bigoplus_{u\in S} {}_xA_u \otimes {}_uB_y \to {}_xC_y. $$

We say that $\boldsymbol{C}$ factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$ if and only if all ${}_x\varphi_y$ are invertible. An enriched category $\boldsymbol{C}$ is called factorizable if it factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$, for some $\boldsymbol{A}$ and $\boldsymbol{B}$.

In Theorem 2.3, our first main result, under the additional assumption that the tensor product on $\boldsymbol{M}$ is distributive over the direct sum, we show that to every $\boldsymbol{M}$-category $\boldsymbol{C}$ that factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$ corresponds a twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$, that is a family $R := \{{}_xR_z^y\}_{x,y,z\in S}$ of morphisms

$$ {}_xR_z^y : {}_xB_y \otimes {}_yA_z \to \bigoplus_{u\in S} {}_xA_u \otimes {}_uB_z $$

which are compatible with the composition and identity maps in $\boldsymbol{A}$ and $\boldsymbol{B}$ in a certain sense.

Trying to associate to a twisting system $R := \{{}_xR_z^y\}_{x,y,z\in S}$ an $\boldsymbol{M}$-category we encountered some difficulties due to the fact that, in general, the image of ${}_xR_z^y$ is too big. Consequently, in this paper we focus on the particular class of twisting systems for which there is a function $|\cdots| : S \times S \times S \to S$ such that the image of ${}_xR_z^y$ is included into ${}_xA_{|xyz|} \otimes {}_{|xyz|}B_z$, for every $x, y, z \in S$. These twisting systems are characterized in Proposition 2.5. A more precise description of them is given in Corollary 2.7, provided that $\boldsymbol{M}$ satisfies an additional condition (†), see §2.6. A similar result is obtained in Corollary 2.9 for a linear monoidal category.

In this way we are led in §2.10 to the definition of simple twisting systems. For such a twisting system R between $\boldsymbol{B}$ and $\boldsymbol{A}$, in Theorem 2.14 we construct an $\boldsymbol{M}$-category $\boldsymbol{A}\otimes_R\boldsymbol{B}$ which factorizes

through $\boldsymbol{A}$ and $\boldsymbol{B}$. Since it generalizes the twisted tensor product of algebras, $\boldsymbol{A}\otimes_R\boldsymbol{B}$ will be called the twisted tensor product of $\boldsymbol{A}$ and $\boldsymbol{B}$.

In the third section we consider the case when $\boldsymbol{M}$ is the monoidal category of coalgebras in a braided category $\boldsymbol{M'}$. In this setting, we prove that there is an one-to-one correspondence between simple twisting systems and matched pair of enriched categories, see §3.6 for the definition of the latter notion. We shall refer to the twisted tensor product of a matched pair as the bicrossed product. By construction, the bicrossed product is a category enriched over $\boldsymbol{M'}$, but we prove that it is enriched over $\boldsymbol{M}$ as well.

More examples of twisted tensor products of enriched categories are given in the last part of the paper. By definition, usual categories are enriched over $\boldsymbol{Set}$, the category of sets. Actually, they are enriched over the monoidal category of coalgebras in $\boldsymbol{Set}$. Hence, simple twisting systems and matched pairs are equivalent notions for usual categories. Moreover, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are thin categories (that is their hom-sets contain at most one morphism), then we show that any twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$ is simple, so it corresponds to a uniquely determined matched pair of categories. We use this result to investigate the twisting systems between two posets.

Our results may be applied to algebras in a monoidal category $\boldsymbol{M}$, which are enriched categories with one object. Therefore, we are also able to recover all bicrossed product constructions that we discussed at the beginning of this introduction.

Finally, we prove that the bicrossed product of two groupoids is also a groupoid, and we give an example of factorizable groupoid with two objects.

1. Preliminaries and notation.

Mainly for fixing the notation and the terminology, in this section we recall the definition of enriched categories, and then we give some example that are useful for our work.

1.1. Monoidal categories. Throughout this paper $(\boldsymbol{M},\otimes,\mathbf{1},a,l,r)$ will denote a monoidal category with associativity constraints $a_{X,Y,Z} : (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z)$ and unit constraints $l_X : \mathbf{1}\otimes X\to X$ and $r_X : X\otimes\mathbf{1}\to X$. The class of objects of $\boldsymbol{M}$ will be denoted by M_0. Mac Lane's Coherence Theorem states that given two parenthesized tensor products of some objects $X_1,\ldots,X_n$ in $\boldsymbol{M}$ (with possible arbitrary insertions of the unit object $\mathbf{1}$) there is a unique morphism between them that can be written as a composition of associativity and unit constraints, and their inverses. Consequently, all these parenthesized tensor products can be identified coherently, and the parenthesis, associativity constraints and unit constraints may be omitted in computations. Henceforth, we shall always ignore them. The identity morphism of an object X in $\boldsymbol{M}$ will be denoted by the same symbol X.

By definition, the tensor product is a functor. In particular, for any morphisms $f' : X'\to Y''$ and $f'' : X''\to Y''$ in $\boldsymbol{M}$ the following equations hold

$$(f'\otimes Y'')\circ(X'\otimes f'') = f'\otimes f'' = (Y'\otimes f'')\circ(f'\otimes X''). \tag{1}$$

If the coproduct of a family $\{X_i\}_{i\in I}$ of objects in $\boldsymbol{M}$ exists, then it will be denoted as a pair $\left(\bigoplus_{i\in I}X_i,\{\sigma_i\}_{i\in I}\right)$, where the maps $\sigma_i : X_i\to\bigoplus_{i\in I}X_i$ are the canonical inclusions.

1.2. The opposite monoidal category. If $(\boldsymbol{M},\otimes,\mathbf{1},a,l,r)$ is a monoidal category, then one constructs the monoidal category $(\boldsymbol{M}^o,\otimes^o,\mathbf{1}^o,a^o,l^o,r^o)$ as follows. By definition, $\boldsymbol{M}^o$ and $\boldsymbol{M}$ share the same objects and identity morphisms. On the other hand, for two objects X,Y in $\boldsymbol{M}$, one takes $\mathrm{Hom}_{\boldsymbol{M}^o}(X,Y) := \mathrm{Hom}_{\boldsymbol{M}}(Y,X)$. The composition of morphisms in $\boldsymbol{M}^o$

$$\bullet : \mathrm{Hom}_{\boldsymbol{M}^o}(Y,Z)\times\mathrm{Hom}_{\boldsymbol{M}^o}(X,Y)\to\mathrm{Hom}_{\boldsymbol{M}^o}(X,Z)$$

is defined by the formula $f\bullet g := g\circ f$, for any $f : Z\to Y$ and $g : Y\to X$ in $\boldsymbol{M}$. The monoidal structure is defined by $X\otimes^o Y = X\otimes Y$ and $\mathbf{1}^o = \mathbf{1}$. The associativity and unit constraints in $\boldsymbol{M}^o$ are given by $a^o_{X,Y,Z} = a^{-1}_{X,Y,Z}$, $l^o = l^{-1}_X$ and $r^o = r^{-1}_X$. If, in addition $\boldsymbol{M}$ is braided monoidal, with braiding $\chi_{X,Y} : X\otimes Y\to Y\otimes X$ then $\boldsymbol{M}^o$ is also braided, with respect to the braiding χ^o defined by $\chi^o_{X,Y} := (\chi_{X,Y})^{-1}$.

Definition 1.3. Let S be a set. We say that a monoidal category $\boldsymbol{M}$ is *S-distributive* if every S-indexed family of objects in $\boldsymbol{M}$ has a coproduct, and the tensor product is distributive to the left and to the right over any such coproduct. More precisely, $\boldsymbol{M}$ is S-distributive if for any family $\{X_i\}_{i\in S}$ the coproduct $(\bigoplus_{i\in S} X_i, \{\sigma_i\}_{i\in S})$ exists and, for an arbitrary object X,

$$(X\otimes(\bigoplus_{i\in S} X_i), \{X\otimes\sigma_i\}_{i\in S}) \qquad \text{and} \qquad ((\bigoplus_{i\in S} X_i)\otimes X, \{\sigma_i\otimes X\}_{i\in S})$$

are the coproducts of $\{X\otimes X_i\}_{i\in S}$ and $\{X_i\otimes X\}_{i\in S}$, respectively. Note that all monoidal categories are S-distributive, provided that S is a singleton (i.e. the cardinal of S is 1).

1.4. Enriched categories. An *enriched category* $\boldsymbol{C}$ over $(\boldsymbol{M},\otimes,\mathbf{1})$, or an $\boldsymbol{M}$-category for short, consists of:

1. A class of objects, that we denote by C_0. If C_0 is a set we say that $\boldsymbol{C}$ is *small*.
2. A hom-object ${}_xC_y$ in $\boldsymbol{M}$, for each x and y in C_0. It plays the same role as $\mathrm{Hom}_{\boldsymbol{C}}(y,x)$, the set of morphisms from y to x in an ordinary category $\boldsymbol{C}$.
3. A morphism ${}_xc_z^y : {}_xC_y\otimes {}_yC_z \to {}_xC_z$, for all x, y and z in C_0.
4. A morphism $1_x : \mathbf{1}\to {}_xC_x$, for all x in C_0.

By definition one assumes that the diagrams in Figure 1 are commutative, for all x, y, z and t in C_0. The commutativity of the square means that the composition of morphisms in $\boldsymbol{C}$, defined by $\{{}_xc_z^y\}_{z,y,z\in C_0}$, is *associative*. We shall say that 1_x is the *identity morphism* of $x\in C_0$.

$$\begin{tikzcd}
{}_xC_y\otimes {}_yC_z\otimes {}_zC_t \arrow[r, "{}_xc_z^y\otimes {}_zC_t"] \arrow[d, "{}_xC_y\otimes {}_yc_t^z"'] & {}_xC_z\otimes {}_zC_t \arrow[d, "{}_xc_t^z"] \\
{}_xC_y\otimes {}_yC_t \arrow[r, "{}_xc_t^y"'] & {}_xC_t
\end{tikzcd}
\qquad
\begin{tikzcd}
{}_xC_y\otimes {}_yC_y \arrow[dr, "{}_xc_y^y"'] & {}_xC_y \arrow[l, "{}_xC_y\otimes 1_y"'] \arrow[r, "1_x\otimes {}_xC_y"] \arrow[d, equal] & {}_xC_x\otimes {}_xC_y \arrow[dl, "{}_xc_y^x"] \\
& {}_xC_y &
\end{tikzcd}$$

Figure 1. The definition of enriched categories.

An $\boldsymbol{M}$-functor $\boldsymbol{\alpha}:\boldsymbol{C}\to\boldsymbol{C'}$ is a couple $(\alpha_0, \{{}_x\alpha_y\}_{x,y\in C_0})$, where $\alpha_0 : C_0\to C_0'$ is a function and ${}_x\alpha_y : {}_xC_y \to {}_{x'}C'_{y'}$ is a morphism in $\boldsymbol{M}$ for any $x,y\in C_0$, where for simplicity we denoted $\alpha_0(u)$ by u', for any $u\in C_0$. By definition, α_0 and ${}_x\alpha_y$ must satisfy the following conditions

$${}_x\alpha_x\circ 1_x^C = 1_{x'}^D \qquad \text{and} \qquad {}_{x'}d_{z'}^{y'}\circ({}_x\alpha_y\otimes {}_y\alpha_z) = {}_x\alpha_z\circ {}_xc_z^y.$$

1.5. To work easier with tensor products of hom-objects in $\boldsymbol{M}$-categories we introduce some new notation. Let S be a set and for every $i=1,\dots,n+1$ we pick up a family $\{{}_xX_y^i\}_{x,y\in S}$ of objects in $\boldsymbol{M}$. If $x_1,\dots,x_{n+1}\in S$ then the tensor product ${}_{x_0}X_{x_1}^1\otimes {}_{x_1}X_{x_2}^2\otimes\cdots\otimes {}_{x_{n-1}}X_{x_n}^n\otimes {}_{x_n}X_{x_{n+1}}^{n+1}$ will be denoted by ${}_{x_0}X_{x_1}^1X_{x_2}^2\cdots {}_{x_n}X_{x_{n+1}}^{n+1}$. Assuming that $\boldsymbol{M}$ is S-distributive and fixing x_0 and x_{n+1}, one can construct inductively the iterated coproduct

$${}_{x_0}X_{\overline{x}_1}^1\cdots {}_{\overline{x}_{n-1}}X_{\overline{x}_n}^n X_{x_{n+1}}^{n+1} := \bigoplus_{x_1\in S}\cdots\bigoplus_{x_n\in S} {}_{x_0}X_{x_1}^1\cdots {}_{x_{n-1}}X_{x_n}^n X_{x_{n+1}}^{n+1}. \tag{2}$$

It is not difficult to see that this object is a coproduct of $\{{}_{x_0}X_{x_1}^1\cdots {}_{x_n}X_{x_{n+1}}^{n+1}\}_{(x_1,\dots,x_n)\in S^n}$. Moreover, as a consequence of the fact that the tensor product is distributive over the direct sum, we have

$${}_{x_0}X_{\overline{x}_1}^1\cdots {}_{\overline{x}_{n-1}}X_{\overline{x}_n}^n X_{x_{n+1}}^{n+1} \cong \bigoplus_{x_1\in S}\cdots\bigoplus_{x_n\in S} {}_{x_0}X_{x_{\pi(1)}}^1 X_{x_{\pi(2)}}^2\cdots {}_{x_{\pi(n)}}X_{x_{n+1}}^{n+1} \tag{3}$$

for any permutation π of the set $\{1,2,\dots,n\}$. The inclusion of ${}_{x_0}X_{x_1}^1\cdots {}_{x_n}X_{x_{n+1}}^{n+1}$ into the coproduct defined in (2) is also inductively constructed as the composition of the following two arrows

$${}_{x_0}X_{x_1}^1\otimes {}_{x_1}X_{x_2}^1\cdots {}_{x_n}X_{x_{n+1}}^{n+1} \longrightarrow {}_{x_0}X_{x_1}^1\otimes {}_{x_1}X_{\overline{x}_2}^1\cdots {}_{\overline{x}_n}X_{x_{n+1}}^{n+1} \hookrightarrow \bigoplus_{x_1\in S} {}_{x_0}X_{x_1}^1\otimes {}_{x_1}X_{\overline{x}_2}^1\cdots {}_{\overline{x}_n}X_{x_{n+1}}^{n+1},$$

where the first morphism is the tensor product between the identity of ${}_{x_0}X_{x_1}^1$ and the inclusion of ${}_{x_1}X_{x_2}^1\cdots {}_{x_n}X_{x_{n+1}}^{n+1}$ into ${}_{x_1}X_{\overline{x}_2}^1\cdots {}_{\overline{x}_n}X_{x_{n+1}}^{n+1}$. Clearly, for every $x_{n+1}\in S$,

$${}_{\overline{x}_0}X_{\overline{x}_1}^1\cdots {}_{\overline{x}_{n-1}}X_{\overline{x}_n}^n X_{x_{n+1}}^{n+1} := \bigoplus_{x_0\in S} {}_{x_0}X_{\overline{x}_1}^1\cdots {}_{\overline{x}_{n-1}}X_{\overline{x}_n}^n X_{x_{n+1}}^{n+1}$$

is the coproduct of $\{{}_{x_0}X^1_{x_1}\cdots{}_{x_n}X^{n+1}_{x_{n+1}}\}_{(x_0,x_1,\ldots,x_n)\in S^n}$. The objects ${}_{x_0}X^1_{\overline{x}_1}\cdots{}_{\overline{x}_{n-1}}X^n_{\overline{x}_n}X^{n+1}_{\overline{x}_{n+1}}$ and ${}_{\overline{x}_0}X^1_{\overline{x}_1}\cdots{}_{\overline{x}_{n-1}}X^n_{\overline{x}_n}X^{n+1}_{\overline{x}_{n+1}}$ are analogously defined.

A similar notation will be used for morphisms. Let us suppose that ${}_x\alpha^i_y$ is a morphism in $\boldsymbol{M}$ with source ${}_xX^i_y$ and target ${}_xY^i_y$, where $x,y\in S$ and $i\in\{1,\ldots,n+1\}$. We set

$$ {}_{x_1}\alpha^1_{x_2}\alpha^2_{x_3}\cdots{}_{x_n}\alpha^{n+1}_{x_{n+1}} := {}_{x_0}\alpha^1_{x_1}\otimes\cdots\otimes{}_{x_n}\alpha^{n+1}_{x_{n+1}}. $$

By the universal property of coproducts, $\{{}_{x_0}\alpha^1_{x_1}\cdots{}_{x_n}\alpha^{n+1}_{x_{n+1}}\}_{(x_1\cdots x_n)\in S^n}$ induces a unique map ${}_{x_0}\alpha^1_{\overline{x}_1}\cdots{}_{\overline{x}_{n-1}}\alpha^{n-1}_{\overline{x}_n}\alpha^n_{x_{n+1}}$ that commutes with the inclusions. In a similar way one constructs

$$ {}_{\overline{x}_0}\alpha^1_{\overline{x}_1}\cdots{}_{\overline{x}_{n-1}}\alpha^n_{\overline{x}_n}\alpha^{n+1}_{x_{n+1}}, \qquad {}_{x_0}\alpha^1_{\overline{x}_1}\cdots{}_{\overline{x}_{n-1}}\alpha^n_{\overline{x}_n}\alpha^{n+1}_{\overline{x}_{n+1}} \qquad \text{and} \qquad {}_{\overline{x}_0}\alpha^1_{\overline{x}_1}\cdots{}_{\overline{x}_{n-1}}\alpha^n_{\overline{x}_n}\alpha^{n+1}_{\overline{x}_{n+1}}. $$

To make the above notation clearer, let us have a look at some examples. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $\boldsymbol{M}$-categories such that $A_0 = B_0 = S$. Recall that the hom-objects in $\boldsymbol{A}$ and $\boldsymbol{B}$ are denoted by ${}_xA_y$ and ${}_xB_y$. Hence, ${}_{\overline{x}}A_y = \bigoplus_{x\in S}{}_xA_y$. We also have ${}_xA_yB_zA_t = {}_xA_y\otimes{}_yB_z\otimes{}_zA_t$ and

$$ {}_xA_{\overline{y}}B_{\overline{z}}A_t = \bigoplus_{y\in S}\bigoplus_{z\in S}{}_xA_yB_zA_t \cong \bigoplus_{z\in S}\bigoplus_{y\in S}{}_xA_yB_zA_t \cong \bigoplus_{y,z\in S}{}_xA_yB_zA_t. $$

Since we have agreed to use the same notation for an object and its identity map, we can write ${}_xB_y\alpha_zA_t\beta_u$ instead of $Id_{{}_xB_y}\otimes{}_y\alpha_z\otimes Id_{{}_zA_t}\otimes{}_t\beta_u$, for any morphisms ${}_y\alpha_z$ and ${}_t\beta_u$ in $\boldsymbol{M}$. The maps ${}_{\overline{x}}a^y_z : {}_{\overline{x}}A_yA_z \longrightarrow {}_{\overline{x}}A_z$ and ${}_xa^y_{\overline{z}} : {}_xA_yA_{\overline{z}} \longrightarrow {}_xA_{\overline{z}}$ are induced by the composition in $\boldsymbol{A}$, that is by the set $\{{}_xa^y_z\}_{z\in S}$. For example, the former map is uniquely defined such that its restriction to ${}_xA_yA_z$ and $\sigma_{x,z}\circ{}_xa^y_z$ coincide for all $x\in S$, where $\sigma_{x,z}$ is the inclusion of ${}_xA_z$ into ${}_{\overline{x}}A_z$. Similarly, ${}_xa^{\overline{y}}_z : {}_xA_{\overline{y}}A_z \longrightarrow {}_xA_z$ is the unique map whose restriction to ${}_xA_yA_z$ is ${}_xa^y_z$, for all $y\in S$.

For more details on enriched categories the reader is referred to [Ke]. We end this section giving some examples of enriched categories.

1.6. The category *Set*. The category of sets is monoidal with respect to the Cartesian product. The unit object is a fixed singleton set, say $\{\emptyset\}$. The coproduct in $\boldsymbol{Set}$ is the disjoint union. Since the disjoint union and the Cartesian product commute, $\boldsymbol{Set}$ is S-distributive for any set S. Clearly, a $\boldsymbol{Set}$-category is an ordinary category. If $\boldsymbol{C}$ is such a category, then an element $f\in{}_xC_y$ will be thought of as a morphism from y to x, and it will be denoted by $f : y\to x$, as usual. In this case we shall say that y (respectively x) is the domain or the source (respectively the codomain or the target) of f. The same notation and terminology will be used for arbitrary $\boldsymbol{M}$-categories, whose objects are sets.

1.7. The category $\mathbb{K}$-*Mod*. Let $\mathbb{K}$ be a commutative ring. The category of $\mathbb{K}$-modules is monoidal with respect to the tensor product of $\mathbb{K}$-modules. The unit object is $\mathbb{K}$, regarded as a $\mathbb{K}$-module. This monoidal category is S-distributive for any S. By definition, a $\mathbb{K}$-*linear category* is an enriched category over $\mathbb{K}$-$\boldsymbol{Mod}$.

1.8. The category Λ-*Mod*-Λ. Let Λ be a $\mathbb{K}$-algebra and let Λ-$\boldsymbol{Mod}$-Λ denote the category of left (or right) modules over $\Lambda\otimes_{\mathbb{K}}\Lambda^o$, where Λ^o is the opposite algebra of Λ. Thus, M is an object in Λ-$\boldsymbol{Mod}$-Λ if, and only if, it is a left and a right Λ-module and these structures are compatible in the sense that

$$ a\cdot m = m\cdot a \qquad \text{and} \qquad (x\cdot m)\cdot y = x\cdot(m\cdot y) $$

for all $a\in\mathbb{K}$, $x,y\in\Lambda$ and $m\in M$. A morphism in Λ-$\boldsymbol{Mod}$-Λ is a map of left and right Λ-modules. The category of Λ-bimodules is monoidal with respect to $(-)\otimes_\Lambda(-)$. The unit object in Λ-$\boldsymbol{Mod}$-Λ is Λ, regarded as a Λ-bimodule. This monoidal category also is S-distributive for any S.

1.9. The category H-*Mod*. Let H be a bialgebra over a commutative ring $\mathbb{K}$. The category of left H-modules is monoidal with respect to $(-)\otimes_{\mathbb{K}}(-)$. If M and N are H-modules, then the H-action on $M\otimes N$ is given by

$$ h\cdot m\otimes n = \sum h_{(1)}\cdot m\otimes h_{(2)}\cdot n. $$

In the above equation we used the Σ-notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. The unit object is $\mathbb{K}$, which is an H-module with the trivial action, induced by the counit of H. This category is S-distributive, for any S. An enriched category over H-***Mod*** is called *H-module category.*

1.10. The category *Comod*-H. Dually, the category of right H-comodules is monoidal with respect to $(-) \otimes_{\mathbb{K}} (-)$. The coaction on and $M \otimes_{\mathbb{K}} N$ is defined by

$$\rho(m \otimes n) = \sum m_{\langle 0\rangle} \otimes n_{\langle 0\rangle} \otimes m_{\langle 1\rangle} n_{\langle 1\rangle},$$

where $\rho(m) = \sum m_{\langle 0\rangle} \otimes n_{\langle 0\rangle}$, and a similar Σ-notation was used for $\rho(n)$. This category is S-distributive, for any set S. By definition, an *H-comodule category* is an enriched category over ***Comod***-H.

1.11. The category $[\boldsymbol{A}, \boldsymbol{A}]$. Let $\boldsymbol{A}$ be a small category, and let $[\boldsymbol{A}, \boldsymbol{A}]$ denote the category of all endofunctors of $\boldsymbol{A}$. Therefore, the objects in $[\boldsymbol{A}, \boldsymbol{A}]$ are functors $F : \boldsymbol{A} \to \boldsymbol{A}$, while the set ${}_F[\boldsymbol{A}, \boldsymbol{A}]_G$ contains all natural transformations $\mu : G \to F$. The composition in this category is the composition of natural transformations. The category $[\boldsymbol{A}, \boldsymbol{A}]$ is monoidal with respect to the composition of functors. If $\mu : F \to G$ and $\mu' : F' \to G'$ are natural transformations, then the natural transformations $\mu F'$ and $G\mu'$ are given by

$$\mu F' : F \circ F' \to G \circ G', \qquad (\mu F')_x := \mu_{F'(x)},$$
$$G\mu' : G \circ F' \to G \circ G', \qquad (G\mu')_x := G(\mu'_x).$$

We can now define the tensor product of μ and μ' by

$$\mu \otimes \mu' := G\mu' \circ \mu F' = \mu G' \circ F\mu'.$$

Even if $\boldsymbol{A}$ is S-distributive, $[\boldsymbol{A}, \boldsymbol{A}]$ may not have this property. In spite of the fact that, by assumption, any S-indexed family in $[\boldsymbol{A}, \boldsymbol{A}]$ has a coproduct, in general this does not commute with the composition of functors. Nevertheless, as we have already noticed, $[\boldsymbol{A}, \boldsymbol{A}]$ is S-distributive if $|S| = 1$.

This remark will allow us to apply our main results to an $[\boldsymbol{A}, \boldsymbol{A}]$-category $\boldsymbol{C}$ with one object x. Hence $F := {}_xC_x$ is an endofunctor of $\boldsymbol{A}$, and the composition and the identity morphisms in $\boldsymbol{C}$ are uniquely defined by natural transformations

$$\mu : F \circ F \to F \qquad \text{and} \qquad \iota : \mathrm{Id}_{\boldsymbol{A}} \to F.$$

The commutativity of the diagrams in Figure 1 is equivalent in this case with the fact that (F, μ, ι) is a *monad*, see [Be] for the definition of monads. In conclusion, monads are in one-to-one correspondence to $[\boldsymbol{A}, \boldsymbol{A}]$-categories with one object.

1.12. The category *Opmon*$(\boldsymbol{M})$. Let $(\boldsymbol{M}, \otimes, 1)$ be a monoidal category. An opmonoidal functor is a triple (F, δ, ε) that consists of

(1) A functor $F : \boldsymbol{M} \to \boldsymbol{M}$.
(2) A natural transformation $\delta := \{\delta_{x,y}\}_{(x,y)\in M_0\times M_0}$, with $\delta_{x,y} : F(x \otimes y) \to F(x) \otimes F(y)$.
(3) A map $\varepsilon : F(\mathbf{1}) \to \mathbf{1}$ in $\boldsymbol{M}$.

In addition, the transformations δ and ε are assumed to render commutative the diagrams in Figure 2. An opmonoidal transformation $\alpha : (F, \delta, \varepsilon) \to (F', \delta', \varepsilon')$ is a natural map $\alpha : F \to F'$ such that, for arbitrary objects x and y in $\boldsymbol{M}$,

$$(\alpha_x \otimes \alpha_y) \circ \delta_{x,y} = \delta'_{x,y} \circ \alpha_{x\otimes y} \qquad \text{and} \qquad \varepsilon' \circ \alpha_{\mathbf{1}} = \varepsilon.$$

Obviously the composition of two opmonoidal transformations is opmonoidal, and the identity of an opmonoidal functor is an opmonoidal transformation. The resulting category will be denoted by ***Opmon***$(\boldsymbol{M})$. For two opmonoidal functors (F, δ, ε) and $(F', \delta', \varepsilon')$ one defines

$$(F, \delta, \varepsilon) \otimes (F', \delta', \varepsilon') := (F \circ F', \delta_{F',F'} \circ F(\delta'), \varepsilon \circ F(\varepsilon')),$$

where $\delta_{F',F'} = \left\{\delta_{F'(x),F'(y)}\right\}_{x,y\in M_0}$. On the other hand, if $\mu : F \to G$ and $\mu'' \to G'$ are opmonoidal transformations, then $\mu \otimes \mu' := \mu G' \circ F\mu'$ is opmonoidal too. One can see easily that $\otimes$ defines a monoidal structure on ***Opmon***$(\boldsymbol{M})$ with unit object $(\mathrm{Id}_{\boldsymbol{M}}, \{\mathrm{Id}_{x\otimes y}\}_{x,y\in M_0}, \mathrm{Id}_{\mathbf{1}})$.

$$\begin{tikzcd} F(x\otimes y\otimes z) \arrow[r, "\delta_{x\otimes y,z}"] \arrow[d, "\delta_{x,y\otimes z}"'] & F(x\otimes y)\otimes F(z) \arrow[d, "\delta_{x,y}\otimes F(z)"] \\ F(x)\otimes F(y\otimes z) \arrow[r, "F(x)\otimes \delta_{y,z}"'] & F(x)\otimes F(y)\otimes F(z) \end{tikzcd} \qquad \begin{tikzcd} F(x)\otimes F(\mathbf{1}) \arrow[r, "F(x)\otimes\varepsilon_{\mathbf{1}}"] & F(x) & F(\mathbf{1})\otimes F(x) \arrow[l, "\varepsilon_{\mathbf{1}}\otimes F(x)"'] \\ & F(x) \arrow[ul, "\delta_{x,\mathbf{1}}"] \arrow[u, equal] \arrow[ur, "\delta_{\mathbf{1},x}"'] & \end{tikzcd}$$

Figure 2. The definition of opmonoidal functors.

1.13. The categories $\boldsymbol{Alg(M)}$ and $\boldsymbol{Coalg(M)}$. Let $(\boldsymbol{M},\otimes,\mathbf{1},\chi)$ be a braided monoidal category with braiding $\chi:=\{\chi_{x,y}\}_{(x,y)\in M_0\times M_0}$, where $\chi_{x,y}:x\otimes y\to y\otimes x$. The category $\boldsymbol{Alg(M)}$ of all algebras in $\boldsymbol{M}$ is monoidal too. Recall that an algebra in $\boldsymbol{M}$ is an $\boldsymbol{M}$-category with one object. As in §1.11, such a category is uniquely determined by an object X in $\boldsymbol{M}$ and two morphisms $m:X\otimes X\to X$ (the multiplication) and $u:\mathbf{1}\to X$ (the unit). The commutativity of the diagrams in Figure 1 means that the algebra is associative and unital. If (X,m,u) and (X',m',u') are algebras in $\boldsymbol{M}$, then $X\otimes X'$ is an algebra in $\boldsymbol{M}$ with multiplication

$$(m\otimes m')\circ(X\otimes\chi_{X',X}\otimes X'):(X\otimes X')\otimes(X\otimes X')\to X\otimes X'$$

and unit $u\otimes u':\mathbf{1}\to X\otimes X'$.

The monoidal category $\boldsymbol{Coalg(M)}$ of coalgebras in $\boldsymbol{M}$ can be defined in a similar way. Alternatively, one may take $\boldsymbol{Coalg(M)}:=\boldsymbol{Alg(M^o)^o}$. Note that the monoidal category of coalgebras in $\boldsymbol{M}$ and the monoidal category of algebras in $\boldsymbol{M}^o$ are opposite each other.

It is not hard to see that $\boldsymbol{Coalg(M)}$ is S-distributive, provided that $\boldsymbol{M}$ is so.

2. Factorizable $\boldsymbol{M}$-categories and twisting systems.

In this section we define factorizable $\boldsymbol{M}$-categories and twisting systems. We shall prove that to every factorizable system corresponds a certain twisting system. Under a mild extra assumption on the monoidal category $\boldsymbol{M}$, we shall also produce enriched categories using a special class of twisting systems that we call simple.

Throughout this section S denotes a fixed set. We assume that all $\boldsymbol{M}$-categories that we work with are small, and that their set of objects is S.

2.1. Factorizable $\boldsymbol{M}$-categories. Let $\boldsymbol{C}$ be a small enriched category over $(\boldsymbol{M},\otimes,\mathbf{1})$. We assume that $\boldsymbol{M}$ is S-distributive. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{M}$-subcategories of $\boldsymbol{C}$. Note that, by assumption, $A_0=B_0=C_0=S$. For x, y and u in S we define

$${}_x\varphi_y^u:{}_xA_uB_y\to{}_xC_y,\qquad {}_x\varphi_y^u:={}_xc_y^u\circ{}_x\alpha_u\beta_y,\tag{4}$$

where $\boldsymbol{\alpha}:\boldsymbol{A}\to\boldsymbol{C}$ and $\boldsymbol{\beta}:\boldsymbol{B}\to\boldsymbol{C}$ denote the corresponding inclusion $\boldsymbol{M}$-functors. By the universal property of coproducts, for every x and y in S, there is ${}_x\varphi_y:{}_xA_{\overline{u}}B_y\to{}_xC_y$ such that

$${}_x\varphi_y\circ{}_x\sigma_y^u={}_x\varphi_y^u,\tag{5}$$

where ${}_x\sigma_y^u$ is the canonical inclusion of ${}_xA_uB_y$ into ${}_xA_{\overline{u}}B_y$. Note that by the universal property of coproducts ${}_x\varphi_y={}_xc_y^{\overline{u}}\circ{}_x\alpha_{\overline{u}}\beta_y$, as we have ${}_x\alpha_{\overline{u}}\beta_y\circ{}_x\sigma_y^u={}_x\tau_y^u\circ{}_x\alpha_u\beta_y$ and ${}_xc_y^{\overline{u}}\circ{}_x\tau_y^u={}_xc_y^u$, where ${}_x\tau_y^u$ denotes the inclusion of ${}_xC_uC_y$ into ${}_xC_{\overline{u}}C_y$.

We shall say that $\boldsymbol{C}$ *factorizes* through $\boldsymbol{A}$ and $\boldsymbol{B}$ if ${}_x\varphi_y$ is an isomorphism, for all x and y in S. By definition, an $\boldsymbol{M}$-category $\boldsymbol{C}$ is *factorizable* if it factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are certain $\boldsymbol{M}$-subcategories of $\boldsymbol{C}$.

2.2. The twisting system associated to a factorizable $\boldsymbol{M}$-category. Let $\boldsymbol{C}$ be an enriched category over a monoidal category $(\boldsymbol{M},\otimes,\mathbf{1})$. We assume that $\boldsymbol{M}$ is S-distributive. The family $R:=\{{}_xR_z^y\}_{x,y,z\in S}$ of morphisms ${}_xR_z^y:{}_yB_yA_z\to{}_xA_{\overline{u}}B_y$ is called a *twisting system* if the four diagrams in Figure 3 are commutative for all x, y, z and t in S.
Let us briefly explain the notation that we used in these diagrams. As a general rule, we omit all subscripts and superscripts denoting elements in S, and which are attached to a morphism. The symbol $\otimes$ is also omitted. For example, a and 1^A (respectively b and 1^B) stand for the suitable composition maps and identity morphisms in $\boldsymbol{A}$ (respectively $\boldsymbol{B}$). The identity morphism of an

$$
\begin{tikzcd}
{}_xB_yB_zA_t \arrow[r, "bI"] \arrow[d, "RI\circ IR"'] & {}_xB_zA_t \arrow[d, "R"] \\
{}_xA_{\overline{v}}B_{\overline{u}}B_t \arrow[r, "Ib"'] & {}_xA_{\overline{v}}B_t
\end{tikzcd}
\quad
\begin{tikzcd}
{}_xB_yA_zA_t \arrow[r, "Ia"] \arrow[d, "IR\circ RI"'] & {}_xB_yA_t \arrow[d, "R"] \\
{}_xA_{\overline{v}}A_{\overline{u}}B_t \arrow[r, "aI"'] & {}_xA_{\overline{u}}B_t
\end{tikzcd}
\quad
\begin{tikzcd}
{}_xA_y \arrow[r, "1^BI"] \arrow[d, "I1^B"'] & {}_xB_xA_y \arrow[d, "R"] \\
{}_xA_yB_y \arrow[r, "\sigma"'] & {}_xA_{\overline{u}}B_y
\end{tikzcd}
\quad
\begin{tikzcd}
{}_xB_y \arrow[r, "I1^A"] \arrow[d, "1^AI"'] & {}_xB_yA_y \arrow[d, "R"] \\
{}_xA_xB_y \arrow[r, "\sigma"'] & {}_xA_{\overline{u}}B_y
\end{tikzcd}
$$

Figure 3. The definition of twisting systems.

object in $\boldsymbol{M}$ is denoted by I. Thus, by $Ia : {}_xB_yA_zA_t \to {}_xB_yA_t$ we mean ${}_xB_y \otimes {}_ya^z_t$. On the other hand, $aI : {}_xA_{\overline{v}}A_{\overline{u}}B_t \to {}_xA_{\overline{u}}B_t$ is a shorthand notation for ${}_xa^{\overline{v}}_{\overline{u}}B_t$, which in turn is the unique map induced by $\{{}_x\sigma^u_t \circ {}_xa^v_uB_t\}_{u,v\in S}$. We shall keep the foregoing notation in all diagrams that we shall work with.

We claim that to every factorizable $\boldsymbol{M}$-category $\boldsymbol{C}$ corresponds a certain twisting system. By definition, the map ${}_x\varphi_y$ constructed in (5) is invertible for all x and y in S. Let ${}_x\psi_y$ denote the inverse of ${}_x\varphi_y$. For x, y and z in S, we can now define

$$
{}_xR^y_z : {}_xB_yA_z \to {}_xA_{\overline{u}}B_z, \quad {}_xR^y_z := {}_x\psi_z \circ {}_xc^y_z \circ {}_x\beta_y\alpha_z. \tag{6}
$$

Theorem 2.3. *If $\boldsymbol{C}$ is a factorizable enriched category over an S-distributive monoidal category $\boldsymbol{M}$, then the maps in (6) define a twisting system.*

Proof. Let us first prove that the first diagram in Figure 3 is commutative. We fix x, y, z and t in S, and we consider the following diagram.

$$
\begin{tikzcd}
{}_xB_yA_{\overline{u}}B_t \arrow[rr, "\beta\alpha\beta"] \arrow[d, "\beta\alpha I"'] \arrow[drr, phantom, "A"] & & {}_xC_yC_{\overline{u}}C_t \arrow[r, "Ic"] \arrow[d, equal] & {}_xC_yC_t \arrow[dddd, "c"] \\
{}_xC_yC_{\overline{u}}B_t \arrow[rr, "II\beta"] \arrow[d, "cI"'] \arrow[drr, phantom, "B"] & & {}_xC_yC_{\overline{u}}C_t \arrow[d, "cI"] & \\
{}_xC_{\overline{u}}B_t \arrow[rr, "I\beta"] \arrow[d, "\psi I"'] \arrow[drr, phantom, "C"] & & {}_xC_{\overline{u}}C_t \arrow[d, equal] \arrow[r, phantom, "F"] & \\
{}_xA_{\overline{v}}B_{\overline{u}}B_t \arrow[r, "\alpha\beta\beta"] \arrow[d, "Ib"'] \arrow[dr, phantom, "D"] & {}_xC_{\overline{v}}C_{\overline{u}}C_t \arrow[r, "cI"] \arrow[d, "Ic"] \arrow[dr, phantom, "E"] & {}_xC_{\overline{u}}C_t \arrow[d, "c"] & \\
{}_xA_{\overline{v}}B_t \arrow[r, "\alpha\beta"'] & {}_xC_{\overline{v}}C_t \arrow[r, "c"'] & {}_xC_t \arrow[r, equal] & {}_xC_t
\end{tikzcd}
$$

Since the tensor product in a monoidal category is a functor, that is in view of (1), we have

$$
{}_xC_yC_u\beta_t \circ {}_x\beta_y\alpha_uB_t = {}_x\beta_y\alpha_u\beta_t, \tag{7}
$$

for any u in S. Hence by the universal property of the coproduct and the construction of the maps ${}_xC_yC_{\overline{u}}\beta_t$, ${}_x\beta_y\alpha_{\overline{u}}B_t$ and ${}_x\beta_y\alpha_{\overline{u}}\beta_t$ we deduce that the relation which is obtained by replacing u with $\overline{u}$ in (7) holds true. This means that the square (A) is commutative. Proceeding similarly one shows that (B) is commutative as well. Furthermore, ${}_xc^{\overline{v}}_{\overline{u}}C_t$, ${}_x\alpha_{\overline{v}}\beta_{\overline{u}}\beta_t$ and ${}_x\psi_{\overline{u}}B_t$ are induced by $\left\{{}_xc^{\overline{v}}_u \otimes {}_uC_t\right\}_{u\in S}$, $\{{}_x\alpha_{\overline{v}}\beta_u \otimes {}_u\beta_t\}_{u\in S}$ and $\{{}_x\psi_u \otimes {}_uB_t\}_{u\in S}$, respectively. Hence their composite $\lambda := {}_xc^{\overline{v}}_{\overline{u}}C_t \circ {}_x\alpha_{\overline{v}}\beta_{\overline{u}}\beta_t \circ {}_x\psi_{\overline{u}}B_t$ is induced by $\{\lambda_u\}_{u\in S}$, where

$$
\lambda_u = \left({}_xc^{\overline{v}}_u \otimes {}_uC_t\right) \circ \left({}_x\alpha_{\overline{v}}\beta_u \otimes {}_u\beta_t\right) \circ \left({}_x\psi_u \otimes {}_uB_t\right) = \left({}_xc^{\overline{v}}_u \circ {}_x\alpha_{\overline{v}}\beta_u \circ {}_x\psi_u\right) \otimes {}_u\beta_t = \left({}_x\varphi_u \circ {}_x\psi_u\right) \otimes {}_u\beta_t.
$$

Since ${}_x\psi_u$ is the inverse of ${}_x\varphi_u$ it follows that $\lambda_u = {}_xC_u\beta_t$, for every $u \in S$. In conclusion

$$
{}_xc^{\overline{v}}_{\overline{u}}C_t \circ {}_x\alpha_{\overline{v}}\beta_{\overline{u}}\beta_t \circ {}_x\psi_{\overline{u}}B_t = {}_xC_{\overline{u}}\beta_t,
$$

so (C) is a commutative square. Since $\boldsymbol{\beta}$ is an $\boldsymbol{M}$-functor it follows that $\{{}_xC_vc^u_t \circ {}_x\alpha_v\beta_u\beta_t\}_{u,v\in S}$ and $\{{}_x\alpha_v\beta_t \circ {}_xA_vb^u_t\}_{u,v\in S}$ are equal. Therefore these families induce the same morphism, that is

$$
{}_xC_{\overline{v}}c^{\overline{u}}_t \circ {}_x\alpha_{\overline{v}}\beta_{\overline{u}}\beta_t = {}_x\alpha_{\overline{v}}\beta_t \circ {}_xA_{\overline{v}}b^{\overline{u}}_t.
$$

Hence (D) is commutative too. Since the composition of morphisms in $\boldsymbol{C}$ is associative, we have

$$ {}_xc_t^{\overline{v}} \circ {}_xC_{\overline{v}}c_t^{\overline{u}} = {}_xc_t^{\overline{u}} \circ {}_xc_{\overline{u}}^{\overline{v}}C_t \qquad \text{and} \qquad {}_xc_t^{y} \circ {}_xC_yc_t^{\overline{u}} = {}_xc_t^{\overline{u}} \circ {}_xc_{\overline{u}}^{y}C_t. $$

These equations imply that (E) and (F) are commutative. Summarizing, we have just proved that all diagrams (A)-(F) are commutative. By diagram chasing it results that the outer square is commutative as well, that is

$$ {}_x\varphi_t \circ {}_xA_{\overline{v}}b_t^{\overline{u}} \circ {}_xR_{\overline{u}}^{y}B_t = {}_xc_t^{y} \circ {}_x\beta_y\varphi_t. $$

Left composing and right composing both sides of this equation by ${}_x\psi_t$ and ${}_xB_yR_t^z$, respectively, yield

$$ \begin{aligned} {}_xA_{\overline{v}}b_t^{\overline{u}} \circ {}_xR_{\overline{u}}^{y}b_t \circ {}_xB_yR_t^z &= {}_x\psi_t \circ {}_xc_t^y \circ {}_x\beta_y\varphi_t \circ {}_xB_yR_t^z \\ &= {}_x\psi_t \circ {}_xc_t^y \circ {}_x\beta_y\varphi_t \circ {}_xB_y\psi_t \circ {}_xB_yc_t^z \circ {}_xB_y\beta_z\alpha_t \\ &= {}_x\psi_t \circ {}_xc_t^y \circ {}_xC_yc_t^z \circ {}_x\beta_y\beta_z\alpha_t, \end{aligned} $$

where for the second and third relations we used the definition of ${}_yR_t^z$ and that ${}_y\varphi_t$ and ${}_y\psi_t$ are inverses each other. On the other hand, the definition of ${}_xR_t^z$, the fact that $\boldsymbol{\beta}$ is a functor and associativity of the composition in $\boldsymbol{C}$ imply the following sequence of identities

$$ \begin{aligned} {}_xR_t^z \circ {}_xb_z^yA_t &= {}_x\psi_t \circ {}_xc_t^z \circ {}_x\beta_z\alpha_t \circ {}_xb_z^yA_t \\ &= {}_x\psi_t \circ {}_xc_t^z \circ {}_xc_z^yC_t \circ {}_x\beta_y\beta_z\alpha_t \\ &= {}_x\psi_t \circ {}_xc_t^y \circ {}_xC_yc_t^z \circ {}_x\beta_y\beta_za_t. \end{aligned} $$

In conclusion, the first diagram in Figure 3 is commutative. Taking into account the definition of ${}_xR_y^x$, the identity ${}_x\beta_x \circ 1_x^B = 1_x$ and the compatibility relation between the composition and the identity morphisms in an enriched category, we get the following sequence of equations

$$ {}_x\varphi_y \circ {}_xR_y^x \circ 1_x^BA_y = {}_x\varphi_y \circ {}_x\psi_y \circ {}_xc_y^x \circ {}_x\beta_x\alpha_y \circ 1_x^BA_y = {}_xc_y^x \circ 1_x\alpha_y = {}_x\alpha_y. $$

Analogously, using the definition of ${}_x\varphi_y$ and the properties of identity morphisms, we get

$$ {}_x\varphi_y \circ {}_x\sigma_y^y \circ {}_xA_y1_y^B = {}_x\varphi_y^y \circ {}_xA_y1_y^B = {}_xc_y^y \circ {}_x\alpha_y\beta_y \circ {}_xA_y1_y^B = {}_xc_y^y \circ {}_x\alpha_y1_y = {}_x\alpha_y. $$

Since ${}_x\varphi_y$ is an isomorphisms, in view of the above computations, it follows that the third diagram is commutative as well. One can prove in a similar way that the remaining two diagrams in Figure 3 are commutative. □

2.4. We have noticed in the introduction that to every twisting system of groups (or, equivalently, every matched pair of groups) one associates a factorizable group. Trying to prove a similar result for a twisting system R between the $\boldsymbol{M}$-categories $\boldsymbol{B}$ and $\boldsymbol{A}$ we have encountered some difficulties due to the fact that, in general, the image of the map

$$ {}_xR_z^y : {}_xB_yA_z \to \bigoplus_{u\in S} {}_xA_uB_z $$

is not included into a summand ${}_xA_uB_z$, for some $u \in S$ that depends on x, y and z. For this reason, in this paper we shall investigate only those twisting systems for which there are a function $|\cdots| : S^3 \to S$ and the maps ${}_x\widetilde{R}_z^y : {}_xB_yA_z \to {}_xA_{|xyz|}B_z$ such that

$$ {}_xR_z^y = {}_x\sigma_z^{|xyz|} \circ {}_x\widetilde{R}_z^y, \tag{8} $$

for all $x, y, z, \in S$. For them we shall use the notation $(\widetilde{R}, |\cdots|)$.

Proposition 2.5. *Let $\boldsymbol{M}$ be a monoidal category which is S-distributive. Let $|\cdots| : S^3 \to S$ and $\{{}_x\widetilde{R}_z^y\}_{x,y,z\in S}$ be a function and a set of maps as above. The family $\{{}_xR_z^y\}_{x,y,z\in S}$ defined by*

(8) is a twisting system if and only if, for any $x, y, z, t \in S$, the following relations hold:

$$ {}_x\sigma_t^{|xy|yzt||}\circ {}_xA_{|xy|yzt||}b_t^{|yzt|}\circ {}_x\widetilde{R}^y_{|yzt|}B_t \circ {}_xB_y\widetilde{R}^z_t = {}_x\sigma_t^{|xzt|}\circ {}_x\widetilde{R}^z_t\circ {}_xb^y_zA_t, \tag{9}$$

$$ {}_x\sigma_t^{|xyz|zt||}\circ {}_xa^{|xyz|}_{||xyz|zt|}B_t\circ {}_xA_{|xyz|}\widetilde{R}^z_t\circ {}_x\widetilde{R}^y_z A_t = {}_x\sigma_t^{|xyt|}\circ {}_x\widetilde{R}^y_t\circ {}_xB_ya^z_t, \tag{10}$$

$$ {}_x\sigma_y^{|xxy|}\circ {}_x\widetilde{R}^x_y\circ(1^B_x\otimes {}_xA_y) = {}_x\sigma^y_y\circ({}_xA_y\otimes 1^B_y), \tag{11}$$

$$ {}_x\sigma_y^{|xyy|}\circ {}_x\widetilde{R}^y_y\circ({}_xB_y\otimes 1^A_y) = {}_x\sigma^x_y\circ(1^A_x\otimes {}_xB_y). \tag{12}$$

Proof. We claim that $\{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$ satisfy (9) if and only if $\{{}_xR^y_z\}_{x,y,z\in S}$ render commutative the first diagram in Figure 3. Indeed, let us consider the following diagram.

$$\begin{tikzcd}
{}_xB_yB_zA_t \arrow[r, "I\widetilde{R}"] \arrow[d, "bI"'] & {}_xB_yA_{|yzt|}B_t \arrow[r, "I\sigma"] \arrow[d, "\widetilde{R}I"'] \arrow[dr, phantom, "B"] & {}_xB_yA_{\overline{u}}B_t \arrow[d, "RI"] \\
{}_xB_zA_t \arrow[dd, "\widetilde{R}"'] & {}_xA_{|xy|yzt||}B_{|yzt|}B_t \arrow[r, "\sigma I"] \arrow[d, "Ib"'] \arrow[dr, phantom, "C"] & {}_xA_{\overline{v}}B_{\overline{u}}B_t \arrow[d, "Ib"] \\
{} \arrow[r, phantom, "A"] & {}_xA_{|xy|yzt||}B_t \arrow[r, "\sigma"] & {}_xA_{\overline{v}}B_t \arrow[d, equal] \\
{}_xA_{|xzt|}B_t \arrow[rr, "\sigma"'] & & {}_xA_{\overline{v}}B_t
\end{tikzcd}$$

The squares (B) and (C) are commutative by the definition of ${}_xR^y_{\overline{u}} : {}_xB_yA_{\overline{u}} \to {}_xA_{\overline{v}}B_{\overline{u}}$ and ${}_{\overline{v}}b^{\overline{u}}_t : {}_{\overline{v}}B_{\overline{u}}B_t \to {}_{\overline{v}}B_t$. Hence the hexagon (A) is commutative if and only if the outer square is commutative. This proves our claim as (A) and the outer square in Figure 3 are commutative if and only if (9) holds and the first diagram in Figure 3 is commutative, respectively. Similarly one shows that the commutativity of the second diagram from Figure 3 is equivalent to (10). On the other hand, obviously, the third and fourth diagrams in Figure 3 are commutative if and only if (11) and (12) hold, so the proposition is proved. □

The inclusion maps make difficult to handle the equations (9)-(12). In some cases we can remove these morphisms by imposing more conditions on the map $|\cdots|$ or on the monoidal category $\boldsymbol{M}$.

2.6. The assumption (†). Let $\boldsymbol{M}$ be a monoidal category which is S-distributive. We shall say that $\boldsymbol{M}$ satisfies the hypothesis (†) if for any coproduct $(\bigoplus_{i\in S}X_i, \{\sigma_i\}_{i\in S})$ in $\boldsymbol{M}$ and any morphisms $f' : X \to X_{i'}$ and $f'' : X \to X_{i''}$ such that $\sigma_{i'}\circ f' = \sigma_{i''}\circ f''$, then either X is an initial object $\emptyset$ in $\boldsymbol{M}$, or $f' = f''$ and $i' = i''$.

The prototype for the class of monoidal categories that satisfy the condition (†) is $\boldsymbol{Set}$. Indeed, let $\{X_i\}_{i\in S}$ be a family of sets, and let σ_i denote the inclusion of X_i into the disjoint union $\coprod_{i\in S} X_i$. We assume that $f' : X \to X_{i'}$ and $f'' : X \to X_{i''}$ are functions such that X in not the empty set, the initial object of $\boldsymbol{Set}$, and $\sigma_{i'}\circ f' = \sigma_{i''}\circ f''$. Then in view of the computation

$$(i', f'(x)) = (\sigma_{i'}\circ f')(x) = (\sigma_{i''}\circ f'')(x) = (i'', f''(x))$$

it follows that $f' = f''$ and $i' = i''$.

Corollary 2.7. *Let $\boldsymbol{M}$ be an S-distributive monoidal category. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $\boldsymbol{M}$-categories such that $A_0 = B_0 = S$. Given a function $|\cdots| : S^3 \to S$ and the maps $\{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$ as in §2.4, let us consider the following four conditions:*

(i) If ${}_xB_yB_zA_t$ is not an initial object, then $|xy|yzt|| = |xzt|$ and

$$ {}_xA_{|xzt|}b_t^{|yzt|}\circ {}_x\widetilde{R}^y_{|yzt|}B_t\circ {}_xB_y\widetilde{R}^z_t = {}_x\widetilde{R}^z_t\circ {}_xb^y_zA_t; \tag{13}$$

(ii) If ${}_xB_yA_zA_t$ is not an initial object, then $||xyz|zt| = |xyt|$ and

$$ {}_xa^{|xyz|}_{|xyt|}B_t\circ {}_xA_{|xyz|}\widetilde{R}^z_t\circ {}_x\widetilde{R}^y_zA_t = {}_x\widetilde{R}^y_t\circ {}_xB_ya^z_t; \tag{14}$$

(iii) If ${}_xA_y$ is not an initial object, then $|xxy| = y$ and

$$ {}_x\widetilde{R}^x_y \circ (1^B_x \otimes {}_xA_y) = {}_xA_y \otimes 1^B_y; \tag{15} $$

(iv) If ${}_xB_y$ is not an initial object, then $|xyy| = x$ and

$$ {}_x\widetilde{R}^y_y \circ ({}_xB_y \otimes 1^A_y) = 1^A_x \otimes {}_xB_y. \tag{16} $$

The above conditions imply the relations (9)-(12). Under the additional assumption that $\boldsymbol{M}$ satisfies the hypothesis (†), the reversed implication holds as well.

Proof. Let us prove that the condition (i) implies the relation (9). In the case when ${}_xB_yB_zA_t = \emptyset$ this is clear, as both sides of (9) are morphisms from an initial object to ${}_xA_{\overline{u}}B_t$. Let us suppose that ${}_xB_yB_zA_t \neq \emptyset$. By composing both sides of (13) with ${}_x\sigma_t^{|xy|yzt||} = {}_x\sigma_t^{|xzt|}$ we get the equation (9). Similarly, the conditions (ii), (iii) and (iv) imply the relations (10), (11) and (12), respectively.

Let us assume that $\boldsymbol{M}$ satisfies the hypothesis (†). We claim that (9) implies (i). If ${}_xB_yB_zA_t$ is not an initial object we take f' and f'' to be the left hand side and the right hand side of (13), respectively. We also set $i' := |xy|yzt||$ and $i'' := |xzt|$. In view of (†), it follows that $f' = f''$ and $i' = i''$, so our claim has been proved. We conclude the proof in the same way. □

2.8. $\mathbb{K}$-linear monoidal categories. Recall that $\boldsymbol{M}$ is $\mathbb{K}$-linear if its hom-sets are $\mathbb{K}$-modules, and both the composition and the tensor product of morphisms are $\mathbb{K}$-bilinear maps. For instance, $\mathbb{K}$-$\boldsymbol{Mod}$, H-$\boldsymbol{Mod}$, $\boldsymbol{Comod}$-H and Λ-$\boldsymbol{Mod}$-Λ are S-distributive linear monoidal categories, for any set S.

Note that the (†) condition fail in a $\mathbb{K}$-linear monoidal category $\boldsymbol{M}$. Indeed let us pick up an object X, which is not an initial object, and a coproduct $(\bigoplus_{i\in S} X_i, \{\sigma_i\}_{i\in S})$ in $\boldsymbol{M}$. If $f' : X \to X_{i'}$ and $f'' : X \to X_{i''}$ are the zero morphisms, then of course $\sigma_{i'} \circ f' = \sigma_{i''} \circ f''$, but neither $i' = i''$ nor $f' = f''$, in general.

Nevertheless, the relations (9)-(12) can also be simplified if $\boldsymbol{M}$ is a linear monoidal category. For any coproduct $(\bigoplus_{i\in S} X_i, \{\sigma_i\}_{i\in S})$ in $\boldsymbol{M}$ and every $i \in S$, there is a map $\pi_i : \bigoplus_{i\in S} X_i \to X_i$ such that $\pi_i \circ \sigma_i = X_i$ and $\pi_i \circ \sigma_j = 0$, provided that $j \neq i$. Hence, supposing that $f' : X \to X_{i'}$ and $f'' : X \to X_{i''}$ are morphisms such that $\sigma_{i'} \circ f' = \sigma_{i''} \circ f''$, we must have either $i' = i''$ and $f' = f''$, or $i' \neq i''$ and $f' = 0 = f''$.

Using the above property of linear monoidal categories, and proceeding as in the proof of the previous corollary, we get the following result.

Corollary 2.9. *Let $\boldsymbol{M}$ be an S-distributive $\mathbb{K}$-linear monoidal category. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{M}$-categories, then the relations (9)-(12) are equivalent to the following conditions:*

(i) If $|xy|yzt|| = |xzt|$ then the relation (13) holds; otherwise, each side of this identity has to be the zero map;
(ii) If $||xyz|zt| = |xyt|$, then the relation (14) holds; otherwise, each side of this identity has to be the zero map;
(iii) If $|xxy| = y$, then the relation (15) holds; otherwise, each side of this identity has to be the zero map;
(iv) If $|xyy| = x$, then the relation (16) holds; otherwise, each side of this identity has to be the zero map.

2.10. Simple twisting systems. The proper context for constructing an enriched category $\boldsymbol{A} \otimes_R \boldsymbol{B}$ out of a special type of twisting system R is provided by Corollary 2.7.

By definition, the couple $(\widetilde{R}, |\cdots|)$ is a *simple twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$* if the function $|\cdots| : S^3 \to S$ and the maps $\{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$ as in §2.4 satisfy the conditions (i)-(iv) in Corollary 2.7. As a part of the definition, we also assume that ${}_xA_{|xyz|}B_z$ is not an initial object whenever ${}_xB_yA_z$ is not so.

The latter technical assumption will be used to prove the associativity of the composition in $\boldsymbol{A} \otimes_R \boldsymbol{B}$, our categorical version of the twisted tensor product of two algebras, which we are going

to define in the next subsection. Note that for ***Set*** this condition is superfluous (if the source of ${}_x\widetilde{R}^y_z$ is not empty, then its target cannot be the empty set).

For a simple twisting system $(\widetilde{R}, |\cdots|)$ we define the maps ${}_xR^y_z$ using the relation (8). By Corollary 2.7 and Proposition 2.5 it follows that $R := \{{}_xR^y_z\}_{x,y,z\in S}$ is a twisting system.

2.11. The category $\boldsymbol{A}\otimes_R\boldsymbol{B}$. For a simple twisting system $(\widetilde{R}, |\cdots|)$ we set

$$(\boldsymbol{A}\otimes_R\boldsymbol{B})_0 := S \quad\text{and}\quad {}_x(\boldsymbol{A}\otimes_R\boldsymbol{B})_y := \bigoplus_{u\in S} {}_xA_u\otimes {}_uB_y = {}_xA_{\overline{u}}B_y.$$

Let us fix three elements x, y and z in S. By definition ${}_xA_{\overline{u}}B_yA_{\overline{v}}B_z := \bigoplus_{u,v\in S} {}_xA_uB_yA_vB_z$, and

$${}_xA_{\overline{u}}B_yA_{\overline{v}}B_z \cong {}_xA_{\overline{u}}B_y\otimes {}_yA_{\overline{v}}B_z$$

as $\boldsymbol{M}$ is S-distributive. Via this identification, the canonical inclusion of ${}_xA_uB_yA_vB_z$ into the coproduct ${}_xA_{\overline{u}}B_yA_{\overline{v}}B_z$ corresponds to ${}_x\sigma^u_y\sigma^v_z = {}_x\sigma^u_y\otimes {}_y\sigma^v_z$. Thus, there is a unique morphism ${}_xc^y_z : {}_xA_{\overline{u}}B_yA_{\overline{v}}B_z \to {}_xA_{\overline{u}}B_z$ such that

$${}_xc^y_z\circ {}_x\sigma^u_y\sigma^v_z = {}_x\sigma^{|uyv|}_z\circ {}_xa_{|uyv|}b_z\circ {}_xA_u\widetilde{R}^y_vB_z,$$

for all $u, v\in S$. Finally, we set $1_x := {}_x\sigma^x_x\circ(1^A_x\otimes 1^B_x)$, and we define

$${}_x\alpha_y := {}_x\sigma^y_y\circ({}_xA_y\otimes 1^B_y) \qquad\text{and}\qquad {}_x\beta_y := {}_x\sigma^x_y\circ(1^A_x\otimes {}_xB_y).$$

2.12. Domains. To show that the above data define an enriched monoidal category $\boldsymbol{A}\otimes_R\boldsymbol{B}$ we need an extra hypothesis on $\boldsymbol{M}$. By definition, a monoidal category $\boldsymbol{M}$ is a *domain* in the case when the tensor product of two objects is an initial object if and only if at least one of them is an initial object. By convention, a monoidal category that has no initial objects is a domain as well.

Obviously ***Set*** is a domain. If $\mathbb{K}$ is a field, then $\mathbb{K}$-***Mod*** is a domain. Keeping the assumption on $\mathbb{K}$, the categories H-***Mod*** and ***Comod***-H are domains, as their tensor product is induced by that one of $\mathbb{K}$-***Mod***. On the other hand, if $\mathbb{K}$ is not a field, then $\mathbb{K}$-***Mod*** and Λ-***Mod***-Λ are not necessarily domains. For instance, $\mathbb{Z}$-***Mod*** $\cong$ $\mathbb{Z}$-***Mod***-$\mathbb{Z}$ is not a domain.

Lemma 2.13. *Let $\boldsymbol{M}$ be an S-distributive monoidal domain. Let $(\widetilde{R}, |\ldots|)$ denote a simple twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$.*

(1) If ${}_xA_uB_yA_vB_zA_wB_t\neq\emptyset$ then $|uyq| = |pvq| = |pzw|$, where $p = |uyv|$ and $q = |vzw|$.
(2) In the following diagram all squares are well defined and commutative.

$$\begin{tikzcd}
{}_xA_uB_yA_vB_zA_wB_t \arrow[r, "III\widetilde{R}I"] \arrow[d, "I\widetilde{R}III"'] \arrow[dr, phantom, "F"] & {}_xA_uB_yA_vA_qB_wB_t \arrow[r, "IIIIb"] \arrow[d, "II\widetilde{R}II\circ I\widetilde{R}III"] \arrow[dr, phantom, "F"] & {}_xA_uB_yA_vA_qB_t \arrow[r, "IIaI"] \arrow[d, "II\widetilde{R}I\circ I\widetilde{R}II"] \arrow[dr, phantom, "R"] & {}_xA_uB_yA_qB_t \arrow[d, "I\widetilde{R}I"] \\
{}_xA_uA_pB_vB_zA_wB_t \arrow[r, bend left, "II\widetilde{R}II\circ III\widetilde{R}I"] \arrow[d, "IIaII"'] \arrow[dr, phantom, "F"] & {}_xA_uA_pA_{|pvq|}B_qB_wB_t \arrow[r, "IIIIb"'] \arrow[d, "aIIII"] \arrow[dr, phantom, "F"] & {}_xA_uA_pA_{|pvq|}B_qB_t \arrow[r, "IaII"'] \arrow[d, "aIII"] \arrow[dr, phantom, "A"] & {}_xA_uA_{|pvq|}B_qB_t \arrow[d, "aII"] \\
{}_xA_pB_vB_zA_wB_t \arrow[r, bend right, "I\widetilde{R}II\circ II\widetilde{R}I"'] \arrow[d, "IbII"'] \arrow[dr, phantom, "L"] & {}_xA_pA_{|pvq|}B_qB_wB_t \arrow[r, "IIIb"'] \arrow[d, "IIbI"] \arrow[dr, phantom, "A"] & {}_xA_pA_{|pvq|}B_qB_t \arrow[r, "aII"'] \arrow[d, "IIb"] \arrow[dr, phantom, "F"] & {}_xA_{|pvq|}B_qB_t \arrow[d, "Ib"] \\
{}_xA_pB_zA_wB_t \arrow[r, "I\widetilde{R}I"'] & {}_xA_pA_{|pvq|}B_wB_t \arrow[r, "IIb"'] & {}_xA_pA_{|pvq|}B_t \arrow[r, "aI"'] & {}_xA_{|pvq|}B_t
\end{tikzcd}$$

Proof. Since $\boldsymbol{M}$ is a domain it follows that any subfactor of ${}_xA_uB_yA_vB_zA_wB_t$ is not an initial object. In particular ${}_vB_zA_w\neq\emptyset$. Thus, by the definition of simple twisting systems, ${}_vA_qB_w$ is not an initial object. In conclusion, ${}_vA_q$ and ${}_qB_w$ are not initial objects in $\boldsymbol{M}$. Since ${}_uB_yA_v\neq\emptyset$ it follows that ${}_uB_yA_vA_q\neq\emptyset$. In view of the definition of simple twisting systems (the second condition) we deduce that $|pvq| = |uyq|$. The other relation can be proved in a similar way.

Let f and g denote the following two morphisms

$$f := {}_xA_ua^p_{|pvq|}B_qB_t\circ {}_xA_uA_p\widetilde{R}^v_qB_t\circ {}_xA_u\widetilde{R}^y_vA_qB_t \quad\text{and}\quad g := {}_xA_u\widetilde{R}^y_qB_t\circ {}_xA_uB_ya^v_qB_t.$$

The target of f is ${}_xA_uA_{|pvq|}B_qB_t$, while the codomain of g is ${}_xA_uA_{|uyq|}B_qB_t$. These two objects may be different for some elements x, u, t, p and q in S. Thus, in general, it does not make sense to speak about the square (R). On the other hand, we have seen that $|uyq| = |pvq|$, if $p = |uyv|$ and $q = |vzw|$. Hence (R) is well defined for these values of p and q. Furthermore, since ${}_uB_yA_vA_q \neq \emptyset$, by definition of simple twisting systems we have

$$ {}_ua^p_{|pvq|}B_q \circ {}_uA_p\widetilde{R}^v_q \circ {}_u\widetilde{R}^y_vA_q = {}_u\widetilde{R}^y_q \circ {}_uB_ya^v_q. \tag{17} $$

By tensoring both sides of the above relation with ${}_xA_u$ on the left and with ${}_qB_t$ on the right we get that $f = g$, i.e. (R) is commutative. Analogously, one shows that (L) is well defined and commutative. All other squares are well defined by construction, their arrows targeting to the right objects. The squares (F) are commutative since the tensor product is a functor. The remaining squares (A) are commutative by associativity. □

Theorem 2.14. *Let $\boldsymbol{M}$ be an S-distributive monoidal domain. If $(\widetilde{R}, |\ldots|)$ is a simple twisting system, then the data in §2.11 define an $\boldsymbol{M}$-category $\boldsymbol{A} \otimes_R \boldsymbol{B}$ that factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$.*

Proof. Let us assume that ${}_xA_uB_yA_vB_zA_wB_t \neq \emptyset$. In view of the previous lemma, the outer square in the diagram from Lemma 2.13 (2) is commutative. It follows that

$$ {}_xc^y_t \circ {}_xA_{\overline{u}}B_yc^z_t \circ {}_x\sigma^u_y\sigma^v_z\sigma^w_t = {}_xc^z_t \circ {}_xc^y_zA_{\overline{w}}B_t \circ {}_x\sigma^u_y\sigma^v_z\sigma^w_t\,. $$

If ${}_xA_uB_yA_vB_zA_wB_t = \emptyset$ this identity obviously holds. Since ${}_xA_{\overline{u}}B_yA_{\overline{v}}B_zA_{\overline{w}}B_t$ is the coproduct of $\{{}_xA_uB_yA_vB_zA_wB_t\}_{u,v,w\in S}$, with the canonical inclusions $\{{}_x\sigma^u_y\sigma^v_z\sigma^w_t\}_{u,v,w\in S}$, we deduce that the composition in $\boldsymbol{A} \otimes_R \boldsymbol{B}$ is associative.

We apply the same strategy to show that $1_x := {}_x\sigma^x_x \circ (1^A_x \otimes 1^B_x)$ is a left identity map of x, that is we have ${}_xc^x_y \circ (1_x \otimes {}_xA_{\overline{u}}B_y) = {}_xA_{\overline{u}}B_y$ for any y. By the universal property of coproducts and the definition of the composition in $\boldsymbol{A} \otimes_R \boldsymbol{B}$, it is enough to prove that

$$ {}_x\sigma^{|xxu|}_y \circ {}_xa^x_{|xxu|}b^u_y \circ {}_xA_x\widetilde{R}^x_uB_y \circ (1^A_x \otimes 1^B_x \otimes {}_xA_uB_y) = {}_x\sigma^u_y, \tag{18} $$

for all $u \in S$. If ${}_xA_u$ is an initial object we have nothing to prove, as the domains of the sides of the above equation are also initial objects (recall that ${}_xA_uB_y = \emptyset$ if ${}_xA_u = \emptyset$). Let us suppose that ${}_xA_u$ is not an initial object. Then by the definition of simple twisting systems (the third condition) we get $|xxu| = u$ and

$$ {}_x\sigma^{|xxu|}_y \circ {}_xa^x_{|xxu|}b^u_y \circ {}_xA_x\widetilde{R}^x_uB_y \circ (1^A_x \otimes 1^B_x \otimes {}_xA_uB_y) = {}_x\sigma^u_y \circ {}_xa^x_ub^u_y \circ (1^A_x \otimes {}_xA_u \otimes 1^B_u \otimes {}_uB_y). $$

Thus the equation (18) immediately follows by the fact 1^A_x and 1^B_u are the identity morphisms of x and u. The fact that 1_x is a right identity map of x can be proved analogously.

We now claim that $\{{}_x\alpha_y\}_{x,y\in S}$ is an $\boldsymbol{M}$-functor. Taking into account the definition of $\boldsymbol{\alpha}$ and ${}_xc^y_z$ we must prove that

$$ {}_x\sigma^{|yyz|}_z \circ {}_xa^x_{|yyz|}a^z_z \circ {}_xA_y\widetilde{R}^y_zA_z \circ ({}_xA_y \otimes 1^A_y \otimes {}_yA_z \otimes 1^A_z) = {}_x\sigma^z_z \circ ({}_xa^y_z \otimes 1^A_z), \tag{19} $$

for all x, y and z in S. Once again, if ${}_yA_z = \emptyset$ we have nothing to prove. In the other case, one can proceed as in the proof of (18) to get this equation. Similarly, $\boldsymbol{\beta}$ is an $\boldsymbol{M}$-functor.

It remains to prove the fact that $\boldsymbol{A} \otimes_R \boldsymbol{B}$ factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$. As a matter of fact, for this enriched category, we shall show that ${}_x\varphi_y$ is the identity map of ${}_x(\boldsymbol{A} \otimes_R \boldsymbol{B})_y$, for all x and y in S. Recall that ${}_x\varphi_y$ is the unique map such that ${}_x\varphi_y \circ {}_x\sigma^u_y = {}_xc^u_y \circ {}_x\alpha_u\beta_y$, for all $u \in S$. Hence to conclude the proof of the theorem it is enough to obtain the following relation

$$ {}_x\sigma^{|uuu|}_y \circ {}_xa^u_{|uuu|}b^u_y \circ {}_xA_u\widetilde{R}^u_uB_y \circ ({}_xA_u \otimes 1^B_u \otimes 1^A_u \otimes {}_uB_z) = {}_x\sigma^u_y, \tag{20} $$

for all $u \in S$. We may suppose that ${}_xA_u$ is not initial object. Thus $|uuu| = u$ and we can take $x = u$ and $y = u$ in (16). Hence, using the same reasoning as in the proof of (18), we deduce the required identity. □

Corollary 2.15. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be enriched categories over an S-distributive monoidal category $\boldsymbol{M}$. Let us suppose that for all x, y, z and t in S the function $|\cdots| : S^3 \to S$ satisfies the equations*

$$ |xy|yzt|| = |xzt|, \qquad ||xyz|zt| = |xyt|, \qquad |xxy| = y \qquad \text{and} \qquad |xyy| = x. \tag{21} $$

If $\{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$ *is a family of maps which satisfies the identities (13)-(16) for all* x, y, z *and* t *in* S, *then the data in §2.11 define an* $\boldsymbol{M}$*-category* $\boldsymbol{A}\otimes_R\boldsymbol{B}$ *that factorizes through* $\boldsymbol{A}$ *and* $\boldsymbol{B}$.

Proof. Let x, y, u, v, z, w and t be arbitrary elements in S. By using the first two identities in (21) we get $|uyq| = |pvq| = |pzw|$, where $p = |uyv|$ and $q = |vzw|$. Hence the first statement in Lemma 2.13 is true. In particular, the squares (R) and (L) in the diagram from Lemma 2.13(2) are well defined. On the other hand, under the assumptions of the corollary, the relation (17) hold. Therefore we can continue as in the proof of the second part of Lemma 2.13 to show that (R) is commutative. Similarly, (L) is commutative too. It follows that the outer square of is commutative too. By the universal property of the coproduct we deduce that the composition is associative, see the first paragraph of the proof of Theorem 2.14.

Furthermore, the relations in (21) together with the identities (13)-(16) imply the equations (18), (19) and (20). Proceeding as in the proof of Theorem 2.14 we conclude that $\boldsymbol{A}\otimes_R\boldsymbol{B}$ is an $\boldsymbol{M}$-category that factorizes through $\boldsymbol{A}$ and $\boldsymbol{B}$. □

Remark 2.16. Throughout this remark we assume that $\boldsymbol{M}$ is a T-distributive monoidal category, where T is an arbitrary set. In other words, any family of objects in $\boldsymbol{M}$ has a coproduct and the tensor product is distributive over all coproducts. It was noticed in [RW, §2.1 and §2.2] that, for such a monoidal category $\boldsymbol{M}$, one can define a bicategory $\boldsymbol{M}$***-mat*** as follows.

By construction, its 0-cells are arbitrary sets. If I and J are two sets, then the 1-cells in $\boldsymbol{M}$***-mat*** from I to J are the $J\times I$-indexed families of objects in $\boldsymbol{M}$. A 2-cell with source $\{X_{ji}\}_{(j,i)\in J\times I}$ and target $\{Y_{ji}\}_{(j,i)\in J\times I}$ is a family $\{f_{ji}\}_{(j,i)\in J\times I}$ of morphisms $f_{ji}: X_{ji}\to Y_{ji}$. The composition of the 1-cells $\{X_{kj}\}_{(k,j)\in K\times J}$ and $\{Y_{ji}\}_{(j,i)\in J\times I}$ is the family $\{Z_{ki}\}_{(k,i)\in K\times I}$, where

$$Z_{ki} := \textstyle\bigoplus_{j\in J} X_{kj}\otimes Y_{ji}.$$

The vertical composition in $\boldsymbol{M}$***-mat*** of $\{f_{ji}\}_{(j,i)\in J\times I}$ and $\{g_{ji}\}_{(j,i)\in J\times I}$ makes sense if and only if the source of f_{ji} and the target of g_{ji} are equal for all i and j. If it exists, then it is defined by

$$\{f_{ji}\}_{(j,i)\in J\times I}\bullet\{g_{ji}\}_{(j,i)\in J\times I} = \{f_{ji}\circ g_{ji}\}_{(j,i)\in J\times I}.$$

Let $\{f_{ji}\}_{(j,i)\in J\times I}$ and $\{f'_{kj}\}_{(k,j)\in K\times J}$ be 2-cells such that $f_{ji}: X_{ji}\to Y_{ji}$ and $f'_{kj}: X'_{kj}\to Y'_{kj}$. By the universal property of coproducts, for each $(k,i)\in K\times I$, there exists a unique morphism $h_{ki}: \bigoplus_{j\in J} X'_{kj}\otimes X_{ji}\to\bigoplus_{j\in J} Y'_{kj}\otimes Y_{ji}$ whose restriction to $X'_{kj}\otimes X_{ji}$ is $f'_{kj}\otimes f_{ji}$. By definition, the horizontal composition of $\{f'_{kj}\}_{(k,j)\in K\times J}$ and $\{f_{ji}\}_{(j,i)\in J\times I}$ is the family $\{h_{ki}\}_{(k,i)\in K\times I}$. The identity 1-cells and 2-cells in $\boldsymbol{M}$***-mat*** are the obvious ones.

As pointed out in [RW], a monad on a set S in $\boldsymbol{M}$***-mat*** is an $\boldsymbol{M}$-category with the set of objects S, and conversely. In particular, given two $\boldsymbol{M}$-categories with the same set of objects, one may speak about distributive laws between the corresponding monads in $\boldsymbol{M}$***-mat***. In our terminology, they are precisely the twisting systems. In view of [RW, §3.1], factorizable enriched categories generalize strict factorization systems.

In conclusion, the Theorem 2.3 may be regarded as a version of [RW, Proposition 3.3] for enriched categories. For a simple twisting system $(\widetilde{R}, |\cdots|)$ between $\boldsymbol{B}$ and $\boldsymbol{A}$, the enriched category $\boldsymbol{A}\otimes_R\boldsymbol{B}$ that we constructed in Theorem 2.14 can also be described in terms of monads. Let $\rho: B\circ A\to A\circ B$ denote the distributive law associated to $(\widetilde{R}, |\cdots|)$, where $(A, m_A, 1_A)$ and $(B, m_B, 1_B)$ are the monads in $\boldsymbol{M}$***-mat*** corresponding to $\boldsymbol{A}$ and $\boldsymbol{B}$**,** respectively. By the general theory of monads in a bicategory, it follows that $A\circ B$ is a monad in $\boldsymbol{M}$***-mat*** with respect to the multiplication and the unit given by the formulae:

$$m := (m_A\circ m_B)\bullet(\mathrm{Id}_A\circ\rho\circ\mathrm{Id}_B) \quad \text{and} \quad 1 := 1_A\circ 1_B.$$

It is not difficult to show that $\boldsymbol{A}\otimes_R\boldsymbol{B}$ is the $\boldsymbol{M}$-category associated to $(A\circ B, m, 1)$.

By replacing ***Set-mat*** with a suitable bicategory, one obtains similar results for other algebraic structures, such as PROs and PROPs; see [La]. We also would like to note that distributive laws between pseudomonads are investigated in [Mar].

We are indebted to the referee for pointing the papers [La, Mar, RW] out to us.

3. Matched pairs of enriched categories.

Throughout this section $(\boldsymbol{M'}, \otimes, \mathbf{1}, \chi)$ denote a braided category and we take $\boldsymbol{M}$ to be the monoidal category $\boldsymbol{Coalg(M')}$. Our aim is to characterize simple twisting systems between two categories that are enriched over $\boldsymbol{M}$. We start by investigating some properties of the morphisms in $\boldsymbol{M}$. For the moment, we impose no conditions on $\boldsymbol{M'}$.

A slightly more general version of the following lemma is stated in [La, Proposition 3.2]. For the sake of completeness we include a proof of it.

Lemma 3.1. *Let $(C, \Delta_C, \varepsilon_C)$, $(D_1, \Delta_{D_1}, \varepsilon_{D_1})$ and $(D_2, \Delta_{D_2}, \varepsilon_{D_2})$ be coalgebras in $\boldsymbol{M'}$. Let $f : C \to D_1 \otimes D_2$ be a morphism of coalgebras. Then $f_1 := (D_1 \otimes \varepsilon_{D_2}) \circ f$ and $f_2 := (\varepsilon_{D_1} \otimes D_2) \circ f$ are coalgebra morphisms and the following relations hold:*

$$(f_1 \otimes f_2) \circ \Delta_C = f, \tag{22}$$

$$(f_2 \otimes f_1) \circ \Delta_C = \chi_{D_1,D_2} \circ (f_1 \otimes f_2) \circ \Delta_C. \tag{23}$$

Conversely, let $f_1 : C \to D_1$ and $f_2 : C \to D_2$ be coalgebra morphisms such that (23) holds. Then $f := (f_1 \otimes f_2) \circ \Delta_C$ is a coalgebra map such that

$$(D_1 \otimes \varepsilon_{D_2}) \circ f = f_1 \quad \text{and} \quad (\varepsilon_{D_1} \otimes D_2) \circ f = f_2. \tag{24}$$

Proof. Let us assume that $f : C \to D_1 \otimes D_2$ is a coalgebra morphism. Let $\varepsilon_i := \varepsilon_{D_i}$, for $i = 1, 2$. Clearly, $D_1 \otimes \varepsilon_{D_2}$ and $\varepsilon_{D_1} \otimes D_2$ are coalgebra morphisms. In conclusion f_1 and f_2 are morphisms in $\boldsymbol{M}$. On the other hand, as f is a morphism in $\boldsymbol{M}$ we have

$$(D_1 \otimes \chi_{D_1,D_2} \otimes D_2) \circ (\Delta_{D_1} \otimes \Delta_{D_2}) \circ f = (f \otimes f) \circ \Delta_C. \tag{25}$$

Hence, using the definition of f_1 and f_2, the relation (25), the fact that the braiding is a natural transformation and the compatibility relation between the comultiplication and the counit we get

$$\begin{aligned}(f_1 \otimes f_2) \circ \Delta_C &= (D_1 \otimes \varepsilon_2 \otimes \varepsilon_1 \otimes D_2) \circ (f \otimes f) \circ \Delta_C \\ &= \big[D_1 \otimes \big((\varepsilon_2 \otimes \varepsilon_1) \circ \chi_{D_1,D_2} \otimes D_2\big) \circ (\Delta_{D_1} \otimes \Delta_{D_2})\big] \circ f \\ &= (D_1 \otimes \varepsilon_1 \otimes \varepsilon_2 \otimes D_2) \circ (\Delta_{D_1} \otimes \Delta_{D_2}) \circ f = f.\end{aligned}$$

By applying $\varepsilon_1 \otimes D_2 \otimes D_1 \otimes \varepsilon_2$ to (25) and using once again the compatibility between the comultiplication and the counit we obtain

$$\begin{aligned}(f_2 \otimes f_1) \circ \Delta_C &= (\varepsilon_1 \otimes D_2 \otimes D_1 \otimes \varepsilon_2) \circ (f \otimes f) \circ \Delta_C \\ &= (\varepsilon_1 \otimes D_2 \otimes D_1 \otimes \varepsilon_2) \circ (D_1 \otimes \chi_{D_1,D_2} \otimes D_2) \circ (\Delta_{D_1} \otimes \Delta_{D_2}) \circ f \\ &= \chi_{D_1,D_2} \circ (\varepsilon_1 \otimes D_1 \otimes D_2 \otimes \varepsilon_2) \circ (\Delta_{D_1} \otimes \Delta_{D_2}) \circ f = \chi_{D_1,D_2} \circ f.\end{aligned}$$

Conversely, let us assume that $f_1 : C \to D_1$ and $f_2 : C \to D_2$ are morphisms in $\boldsymbol{M}$ such that (23) holds. Let $f := (f_1 \otimes f_2) \circ \Delta_C$. By the definition of the comultiplication on $D_1 \otimes D_2$ and the fact that f_1 and f_2 are morphisms in $\boldsymbol{M}$, we get

$$\begin{aligned}\Delta_{D_1 \otimes D_2} \circ f &= (D_1 \otimes \chi_{D_1,D_2} \otimes D_2) \circ (\Delta_{D_1} \otimes \Delta_{D_2}) \circ (f_1 \otimes f_2) \circ \Delta_C \\ &= (D_1 \otimes \chi_{D_1,D_2} \otimes D_2) \circ (f_1 \otimes f_1 \otimes f_2 \otimes f_2) \circ (\Delta_C \otimes \Delta_C) \circ \Delta_C.\end{aligned}$$

Taking into account (23) and the fact that the comultiplication is coassociative, it follows that

$$\begin{aligned}\Delta_{D_1 \otimes D_2} \circ f &= [f_1 \otimes (\chi_{D_1,D_2} \circ (f_1 \otimes f_2) \circ \Delta_C) \otimes f_2] \circ (C \otimes \Delta_C) \circ \Delta_C \\ &= [f_1 \otimes ((f_2 \otimes f_1) \circ \Delta_C) \otimes f_2] \circ (C \otimes \Delta_C) \circ \Delta_C \\ &= [((f_1 \otimes f_2) \circ \Delta_C) \otimes ((f_1 \otimes f_2) \circ \Delta_C)] \circ \Delta_C = (f \otimes f) \circ \Delta_C.\end{aligned}$$

The formula that defines f together with $\varepsilon_i \circ f_i = \varepsilon_C$ yield

$$(\varepsilon_1 \otimes \varepsilon_2) \circ f = (\varepsilon_1 \circ f_1 \otimes \varepsilon_2 \circ f_2) \circ \Delta_C = (\varepsilon_C \otimes \varepsilon_C) \circ \Delta_C = \varepsilon_C.$$

Thus f is a morphism of coalgebras, so the lemma is proved. The equations in (24) are obvious, as $\varepsilon_i \circ f_i = \varepsilon_C$. $\square$

Remark 3.2. Let $f', f'' : C \to D_1 \otimes D_2$ be coalgebra morphisms. By the preceding lemma, f' and f'' are equal if and only if

$$(\varepsilon_1 \otimes D_2) \circ f' = (\varepsilon_1 \otimes D_2) \circ f'' \quad \text{and} \quad (D_1 \otimes \varepsilon_2) \circ f' = (D_1 \otimes \varepsilon_2) \circ f''.$$

3.3. The morphisms ${}_x\triangleright^y_z$ and ${}_x\triangleleft^y_z$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote two $\boldsymbol{M}$-categories whose objects are the elements of a set S. The hom-objects of $\boldsymbol{A}$ and $\boldsymbol{B}$ are coalgebras, which will be denoted by $({}_xA_y, {}_x\Delta^A_y, {}_x\varepsilon^A_y)$ and $({}_xB_y, {}_x\Delta^B_y, {}_x\varepsilon^B_y)$. By definition, the composition and the identity maps in $\boldsymbol{A}$ and $\boldsymbol{B}$ are coalgebra morphisms. Note that the comultiplication of ${}_xB_yA_z$ is given by

$$\Delta_{{}_xB_yA_z} = ({}_xB_y \otimes \chi_{{}_xB_y,{}_yA_z} \otimes {}_yA_z) \circ {}_x\Delta^B_y\Delta^A_z.$$

Let $|\cdots| : S^3 \to S$ be a function and let $\widetilde{R}$ denote an S^3-indexed family of coalgebra morphisms ${}_x\widetilde{R}^y_z : {}_xB_yA_z \to {}_xA_{|xyz|}B_z$. We define ${}_x\triangleright^y_z : {}_xB_yA_z \to {}_xA_{|xyz|}$ and ${}_x\triangleleft^y_z : {}_xB_yA_z \to {}_{|xyz|}B_z$ by

$$ {}_x\triangleright^y_z := {}_xA_{|xyz|}\varepsilon^B_z \circ {}_x\widetilde{R}^y_z \quad \text{and} \quad {}_x\triangleleft^y_z := ({}_x\varepsilon^A_{|xyz|}B_z) \circ {}_x\widetilde{R}^y_z. \tag{26}$$

In view of Lemma 3.1, ${}_x\triangleright^y_z$ and ${}_x\triangleleft^y_z$ are coalgebra morphisms and they satisfy the relations

$$({}_x\triangleright^y_z \otimes {}_x\triangleleft^y_z) \circ \Delta_{{}_xB_yA_z} = {}_x\widetilde{R}^y_z, \tag{27}$$

$$({}_x\triangleleft^y_z \otimes {}_x\triangleright^y_z) \circ \Delta_{{}_xB_yA_z} = \chi_{{}_xA_{|xyz|},{}_{|xyz|}B_z} \circ ({}_x\triangleright^y_z \otimes {}_x\triangleleft^y_z) \circ \Delta_{{}_xB_yA_z}. \tag{28}$$

Conversely, if one starts with $\triangleright := \{{}_x\triangleright^y_z\}_{x,y,z\in S}$ and $\triangleleft := \{{}_x\triangleleft^y_z\}_{x,y,z\in S}$, two families of coalgebra maps that satisfy (28), then by formula (27) we get the set $\widetilde{R} := \{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$ whose elements are coalgebra maps, cf. Lemma 3.1. Therefore, there is an one-to-one correspondence between the couples $(\triangleright, \triangleleft)$ and the sets $\widetilde{R}$ as above. Our goal is to characterize those couples $(\triangleright, \triangleleft)$ that corresponds to a simple twisting system in $\boldsymbol{M'}$.

Lemma 3.4. *The statements below are true.*

(1) *If $|xy\,|yzt|| = |xzt|$ then the relation (13) is equivalent to the following equations:*

$$ {}_x\triangleleft^z_t \circ {}_xb^y_zA_t = {}_{|xzt|}b^{|yzt|}_t \circ {}_x\triangleleft^y_{|yzt|}B_t \circ ({}_xB_y \otimes {}_y\triangleright^z_t \otimes {}_y\triangleleft^z_t) \circ ({}_xB_y \otimes \Delta_{{}_yB_zA_t}), \tag{29}$$

$$ {}_x\triangleright^z_t \circ {}_xb^y_zA_t = {}_x\triangleright^y_{|yzt|} \circ {}_xB_y\triangleright^z_t. \tag{30}$$

(2) *If $|xyz\,|zt|| = |xyt|$ then the relation (14) is equivalent to the following equations:*

$$ {}_x\triangleright^y_t \circ {}_xB_ya^z_t = {}_xa^{|xyz|}_{|xyt|} \circ {}_xA_{|xyz|}\triangleright^z_t \circ ({}_x\triangleright^y_z \otimes {}_x\triangleleft^y_z \otimes {}_zA_t) \circ (\Delta_{{}_xB_yA_z} \otimes {}_zA_t), \tag{31}$$

$$ {}_x\triangleleft^y_t \circ {}_xB_ya^z_t = {}_{|xyz|}\triangleleft^z_t \circ {}_x\triangleleft^y_zA_t. \tag{32}$$

(3) *If $|xyy| = x$ then the relation (15) is equivalent to the following equations:*

$$ {}_x\triangleleft^x_y \circ \left(1^B_x \otimes {}_xA_y\right) = {}_x\varepsilon^A_y \otimes 1^B_y, \tag{33}$$

$$ {}_x\triangleright^x_y \circ \left(1^B_x \otimes {}_xA_y\right) = {}_xA_y. \tag{34}$$

(4) *If $|xxy| = y$ then the relation (16) is equivalent to the following equations:*

$$ {}_x\triangleleft^y_y \circ \left({}_xB_y \otimes 1^A_y\right) = {}_xB_y, \tag{35}$$

$$ {}_x\triangleright^y_y \circ \left({}_xB_y \otimes 1^A_y\right) = 1^A_x \otimes {}_x\varepsilon^B_y. \tag{36}$$

Proof. In order to prove the first statement we apply the Remark 3.2 to

$$f' := {}_x\widetilde{R}^z_t \circ {}_xb^y_zA_t \quad \text{and} \quad f'' := {}_xA_{|xzt|}b^{|yzt|}_t \circ {}_x\widetilde{R}^y_{|yzt|}B_t \circ {}_xB_y\widetilde{R}^z_t.$$

Note that f'' is well defined and its target is ${}_xA_{|xzt|}B_t$, since the codomain of ${}_x\widetilde{R}^y_{|yzt|}B_t \circ {}_xB_y\widetilde{R}^z_t$ is ${}_xA_{|xy|yzt||}B_t$ and $|xy\,|yzt|| = |xzt|$. Clearly, f' and f'' are coalgebra morphisms, since the

composite and the tensor product of two morphisms in $\boldsymbol{M}$ remain in $\boldsymbol{M}$. An easy computation, based on the equation (27) and the formulae of ${}_x\triangleright_z^y$ and ${}_x\triangleleft_z^y$, yields us

$$\begin{aligned}
&{}_x\varepsilon^A_{|xzt|}B_t \circ f' = {}_x\triangleleft_t^z \circ {}_xb_z^yA_t,\\
&{}_xA_{|xzt|}\varepsilon^B_t \circ f' = {}_x\triangleright_t^z \circ {}_xb_z^yA_t,\\
&{}_x\varepsilon^A_{|xzt|}B_t \circ f'' = {}_{|xzt|}b_t^{|yzt|} \circ {}_x\triangleleft^y_{|yzt|} B_t \circ ({}_xB_y \otimes {}_y\triangleright_t^z \otimes_y \triangleleft_t^z) \circ ({}_xB_y \otimes \Delta_{{}_yB_zA_t}).
\end{aligned}$$

Taking into account that ${}_xb_z^y$ is a coalgebra morphism and using the definition of ${}_x\triangleright_z^y$ we get

$$ {}_xA_{|xzt|}\varepsilon^B_t \circ f'' = {}_xA_{|xzt|}\varepsilon^B_{|yzt|} \circ {}_x\widetilde{R}^y_{|yzt|} \circ \Big({}_xB_y \otimes ({}_yA_{|yzt|}\varepsilon^B_t \circ {}_y\widetilde{R}^z_t)\Big) = {}_x\triangleright^y_{|yzt|} \circ_x B_y \triangleright_t^z \,. $$

In view of the Remark 3.2, we have $f' = f''$ if and only if

$$ {}_xA_{|xzt|}\varepsilon^B_t \circ f' = {}_xA_{|xzt|}\varepsilon^B_t \circ f'' \quad \text{and} \quad {}_x\varepsilon^A_{|xzt|}B_t \circ f' = {}_x\varepsilon^A_{|xzt|}B_t \circ f''. $$

Thus, if $|xy\,|yzt|| = |xzt|$, then (13) is equivalent to (29) together with (30). We omit the proof of the second statement, being similar.

To prove the third part of the lemma we reiterate the above reasoning. We now take f' and f'' to be the coalgebra morphisms

$$ f' := {}_x\widetilde{R}^x_y \circ \big(1^B_x \otimes {}_xA_y\big) \quad \text{and} \quad f'' := {}_xA_y \otimes 1^B_y. $$

Since $|xxy| = y$ both $f'$and f'' target in ${}_xA_yB_y$. It is easy to see that (33) together with (34) are equivalent to (15). Similarly, one shows that the fourth statement is true. □

Theorem 3.5. *We keep the notation and the assumptions from §3.3. The set $\widetilde{R}$ is a simple twisting system in $\boldsymbol{M'}$ if and only if the families $\triangleright$ and $\triangleleft$ satisfy the following conditions:*

(i) If ${}_xB_yA_z$ is not an initial object then ${}_xA_{|xyz|}B_z$ is not an initial object as well.
(ii) If ${}_xB_yB_zA_t$ is not an initial object in $\boldsymbol{M'}$, then $|xy|yzt|| = |xzt|$ and the equations (29) and (30) hold.
(iii) If ${}_xB_yA_zA_t$ is not an initial object in $\boldsymbol{M'}$, then $||xyz|zt| = |xyz|$ and the equations (31) and (32) hold.
(iv) If ${}_xA_y$ is not an initial object in $\boldsymbol{M'}$, then $|xxy| = y$ and the equations (33) and (34) hold.
(v) If ${}_xB_y$ is not an initial object in $\boldsymbol{M'}$, then $|xyy| = x$ and the equations (35) and (36) hold.

Proof. The condition (i) is a part of the definition of simple twisting systems. If ${}_xB_yB_zA_t$ is not an initial object in $\boldsymbol{M'}$ then we may assume that $|xy|yzt|| = |xzt|$. Thus, by Lemma 3.4, the relation (13) and the equations (29) and (30) are equivalent. To conclude the proof we proceed in a similar way. □

3.6. Matched pairs and the bicrossed product. Let $\triangleright := \{{}_x\triangleright_z^y\}_{x,y,z\in S}$ and $\triangleleft := \{{}_x\triangleleft_z^y\}_{x,y,z\in S}$ be two families of maps as in §3.3. We shall say that the quintuple $(\boldsymbol{A}, \boldsymbol{B}, \triangleright, \triangleleft, |\cdots|)$ is a *matched pair* of $\boldsymbol{M}$-categories if and only if $\triangleright$ and $\triangleleft$ satisfy the conditions (i)-(v) from the above theorem. For a matched pair $(\boldsymbol{A}, \boldsymbol{B}, \triangleright, \triangleleft, |\cdots|)$ we have just seen that $(\widetilde{R}, |\cdots|)$ is a simple twisting system in $\boldsymbol{M'}$, where $\widetilde{R} := \{{}_x\widetilde{R}_z^y\}_{x,y,z\in S}$ is the set of coalgebra morphisms which are defined by the formula (27). Hence, supposing that $\boldsymbol{M'}$ is an S-distributive domain, we may construct the twisted tensor product $\boldsymbol{A} \otimes_R \boldsymbol{B}$, which is an enriched category over $\boldsymbol{M'}$. We shall call it the *bicrossed product* of $(\boldsymbol{A}, \boldsymbol{B}, \triangleright, \triangleleft, |\cdots|)$ and we shall denote it by $\boldsymbol{A} \bowtie \boldsymbol{B}$.

Proposition 3.7. *The bicrossed product of a matched pair $(\boldsymbol{A}, \boldsymbol{B}, \triangleright, \triangleleft, |\cdots|)$ is enriched over the monoidal category $\boldsymbol{M} := \boldsymbol{Coalg(M')}$.*

Proof. Let $\{C_i\}_{i\in i}$ be a family of coalgebras in $\boldsymbol{M'}$**.** Let us assume that the underlying family of objects has a coproduct $C := \bigoplus_{x\in S} C_i$ in $\boldsymbol{M'}$. Let $\{\sigma_i\}_{i\in I}$ by the set of canonical inclusions into C. There are unique maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbf{1}$ such that

$$ \Delta \circ \sigma_i = (\sigma_i \otimes \sigma_i) \circ \Delta_i \quad \text{and} \quad \varepsilon \circ \sigma_i = \varepsilon_i, $$

for all $i \in I$, where Δ_i and ε_i are the comultiplication and the counit of C_i. It is easy to see that (C, Δ, ε) is a coalgebra in $\boldsymbol{M'}$. Note that, by the construction of the coalgebra structure on C, the inclusion σ_i is a coalgebra map, for any $i \in I$. Furthermore, let $f_i : C_i \to D$ be a coalgebra morphism for every $i \in I$. By the universal property of the coproduct there is a unique map $f : C \to D$ in $\boldsymbol{M'}$ such that $f \circ \sigma_i = f_i$, for all $i \in I$. It is not difficult to see that f is a morphism of coalgebras, so $(C, \{\sigma_i\}_{i\in I})$ is the coproduct of $\{C_i\}_{i\in I}$ in $\boldsymbol{M}$.

In particular, ${}_xA_{\overline{u}}B_y = \bigoplus_{u\in S} {}_xA_{\overline{u}}B_y$ has a unique coalgebra structure such that the inclusion ${}_x\sigma^u_z : {}_xA_uB_y \to {}_xA_{\overline{u}}B_y$ is a coalgebra map, for all x, y and u in S. Recall that for the construction of the composition map ${}_xc^y_z : {}_xA_{\overline{u}}B_yA_{\overline{v}}B_z \to {}_xA_{\overline{w}}B_z$ one applies the universal property of the coproduct to $\{f_{u,v}\}_{u,v\in S^2}$, where

$$f_{u,v} = {}_x\sigma_z^{|uyv|} \circ {}_xa_{|uyv|}b_z \circ {}_xA_u\widetilde{R}^y_vB_z.$$

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{M}$-categories and ${}_u\widetilde{R}^y_v$ is a coalgebra map, in view of the foregoing remarks, it follows that ${}_xc^y_z$ is a morphism in $\boldsymbol{M}$, for all $x, y, z \in S$. The identity map of x in $\boldsymbol{A} \bowtie \boldsymbol{B}$ is the coalgebra map ${}_x\sigma^x_x \circ (1^A_x \otimes 1^B_x)$. In conclusion, $\boldsymbol{A} \bowtie \boldsymbol{B}$ is enriched over $\boldsymbol{M}$. □

4. Examples.

In this section we give some examples of (simple) twisting systems. We start by considering the case of $\boldsymbol{Set}$-categories, that is usual categories.

4.1. Simple twisting systems of enriched categories over $\boldsymbol{Set}$. The category $\boldsymbol{Set}$ is a braided monoidal category with respect to the cartesian product, its unit object being $\{\emptyset\}$. Clearly, the empty set is the initial object in $\boldsymbol{Set}$, and this category is an S-distributive domain, for any set S. We have already noticed that the (†) hypothesis holds in $\boldsymbol{Set}$.

Let $\boldsymbol{C}$ be an enriched category over $\boldsymbol{Set}$. Thus, by definition, $\boldsymbol{C}$ is a category in the usual sense, that is ${}_xC_y$ is a set for all $x, y \in S$. An element in ${}_xC_y$ is regarded as a function from y to x.

It is easy to see that a given set X can be seen in a unique way as a coalgebra in $\boldsymbol{Set}$. As a matter of fact the comultiplication and the counit of this coalgebra are given by the diagonal map $\Delta : X \to X \times X$ and the constant map $\varepsilon : X \to \{\emptyset\}$,

$$\Delta(x) = x \otimes x \quad \text{and} \quad \varepsilon(x) = \emptyset.$$

Obviously, any function $f : X \to Y$ is morphism of coalgebras in $\boldsymbol{Set}$. Consequently, any category $\boldsymbol{C}$ may be seen as an enriched category over $\boldsymbol{Coalg(Set)}$,

Our aim is to describe the simple twisting systems between two categories $\boldsymbol{B}$ and $\boldsymbol{A}$. In view of the foregoing discussion and of our results in the previous section, for any simple twisting system $R := \{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$ there is a unique matched pair $(\boldsymbol{A}, \boldsymbol{B}, \triangleright, \triangleleft, |\cdots|)$, and conversely. These structures are related each other by the formulae (26) and (27).

Since $\boldsymbol{A}$ is an usual category, the composition of morphisms will be denoted in the traditional way $g \circ g'$, for any $g \in {}_xA_y$ and $g' \in {}_yA_z$ (recall that the domain and the codomain of g are y and x, respectively). The same notation will be used for $\boldsymbol{B}$. On the other hand, for any $f \in {}_xB_y$ and $g \in {}_yA_z$ we shall write

$$f \triangleright g := {}_x\triangleright^y_z(f, g) \quad \text{and} \quad f \triangleleft g = {}_x\triangleleft^y_z(f, g).$$

Since the comultiplication in this case is always the diagonal map, and the counit is the constant map to $\{\emptyset\}$, the conditions of Theorem 3.5 and the following ones are equivalent.

(i) If ${}_xB_yA_z$ is not empty then ${}_xA_{|xyz|}B_z$ is not empty as well.

(ii) For any $(f, f', g) \in {}_xB_yB_zA_t$ we have $|xy|yzt|| = |xzt|$, and

$$(f \circ f') \triangleright g = f \triangleright (f' \triangleright g) \qquad \text{and} \qquad (f \circ f') \triangleleft g = [f \triangleleft (f' \triangleright g)] \circ (f' \triangleleft g).$$

(iii) For any $(f, g, g') \in {}_xB_yA_zA_t$ we have $||xyz|zt| = |xyt|$, and

$$f \triangleleft (g \circ g') = (f \triangleleft g) \triangleleft g' \qquad \text{and} \qquad f \triangleright (g \circ g') = (f \triangleright g) \circ [(f \triangleleft g) \triangleright g'].$$

(iv) For any $g \in {}_xA_y$ we have $|xxy| = y$, and

$$1^B_x \triangleright g = g \qquad \text{and} \qquad 1^B_x \triangleleft g = 1^B_y.$$

(v) For any $f\in {}_xB_y$ we have $|xyy|=x$, and

$$f\triangleright 1^A_y=1^B_x \qquad\text{and}\qquad f\triangleleft 1^A_y=f.$$

In this case the bicrossed product $\boldsymbol{A}\bowtie\boldsymbol{B}$ is the category whose hom-sets are given by

$${}_x(\boldsymbol{A}\bowtie\boldsymbol{B})_y=\coprod_{u\in S}{}_xA_uB_y.$$

The identity of x in $\boldsymbol{A}\bowtie\boldsymbol{B}$ is $(1^A_x,1^B_x)$. For $(g,f)\in {}_xA_uB_y$ and $(g',f')\in {}_yA_vB_z$ we have

$$(g,f)\circ(g',f')=(g\circ(f\triangleright g'),(f\triangleleft g')\circ f')\,.$$

Remark 4.2. R. Resebrugh and R.J. Wood showed that every twisting systems between two $\boldsymbol{Set}$-categories $\boldsymbol{B}$ and $\boldsymbol{A}$ is completely determined by a left action $\triangleright$ of $\boldsymbol{B}$ on $\boldsymbol{A}$ and a right action $\triangleleft$ of $\boldsymbol{A}$ on $\boldsymbol{B}$. More precisely, given a twisting system $R=\{{}_xR^y_z\}_{x,y,z\in S}$ and the morphisms $f\in {}_xB_y$ and $g\in {}_yA_z$, then ${}_xR^y_z(f,g)$ is an element in ${}_xA_uB_z$, where u is a certain element of S. Hence, there are unique morphisms $f\triangleright g\in {}_xA_u$ and $f\triangleleft g\in {}_uB_z$ such that

$${}_xR^y_z(g,f)=(f\triangleright g,f\triangleleft g)\,.$$

The actions $\triangleright$ and $\triangleleft$ must satisfy several compatibility conditions, which are similar to those that appear in the above characterization of simple twisting systems. For details the reader is referred to the second section of [RW].

4.3. The bicrossed product of two groupoids. We now assume that $(\boldsymbol{A},\boldsymbol{B},\triangleright,\triangleleft,|\cdots|)$ is a matched pair of groupoids. Recall that a groupoid is a category whose morphisms are invertible. We claim that $\boldsymbol{A}\bowtie\boldsymbol{B}$ is also a groupoid. Indeed, as in the case of monoids, one can show that a category is a groupoid if and only if every morphism is right invertible (or left invertible). Since

$${}_x(\boldsymbol{A}\bowtie\boldsymbol{B})_y=\coprod_{u\in S}{}_xA_uB_y,$$

it is enough to prove that (g,f) is right invertible, where $g\in {}_xA_u$ and $f\in {}_uB_y$ are arbitrary morphisms. Therefore, we are looking for a pair $(g',f')\in {}_yA_v\times {}_vB_x$ such that

$$g\circ(f\triangleright g')=1^A_x \qquad\text{and}\qquad (f\triangleleft g')\circ f'=1^B_x.$$

Since g is an invertible morphism in ${}_xA_u$ we get that $f\triangleright g'=g^{-1}\in {}_uA_x$. Since f is invertible too,

$$g'=1^A_y\triangleright g'=(f^{-1}\circ f)\triangleright g'=f^{-1}\triangleright(f\triangleright g')=f^{-1}\triangleright g^{-1}.$$

As $g'\in {}_yA_v$ and $f^{-1}\triangleright g^{-1}\in {}_yA_{|yux|}$ we must have $v=|yux|$. Thus we can now take

$$f'=[f\triangleleft(f^{-1}\triangleright g^{-1})]^{-1}\in {}_{|yux|}B_x.$$

4.4. The smash product category. We take $\boldsymbol{M}$ to be the monoidal category $\mathbb{K}$-$\boldsymbol{Mod}$, where $\mathbb{K}$ is a commutative ring. Hence in this case we work with $\mathbb{K}$-linear categories. Let H be a $\mathbb{K}$-bialgebra. We define an enriched category $\boldsymbol{H}$ over $\mathbb{K}$-$\boldsymbol{Mod}$ by setting ${}_xH_x=H$ and ${}_xH_y=0$, for $x\neq y$. The composition of morphisms in $\boldsymbol{H}$ is given by the multiplication in H and the identity of x is the unit of H. For the comultiplication of H we shall use the Σ-notation

$$\Delta(h)=\sum h_{(1)}\otimes h_{(2)}.$$

Let $\boldsymbol{A}$ denote an H-module category, i.e. a category enriched in H-$\boldsymbol{Mod}$. Thus H acts on ${}_xA_y$, for any $x,y\in S$, and the composition and the identity maps in $\boldsymbol{A}$ are H-linear morphisms. Obviously, $\boldsymbol{A}$ is a $\mathbb{K}$-linear category. Our aim is to associate to $\boldsymbol{A}$ a simple twisting system $R=\{{}_x\widetilde{R}^y_z\}_{x,y,z\in S}$. First we define $|\cdots|:S^3\to S$ by $|xyz|=z$. Then, using the actions $\cdot: H\otimes {}_xA_z\to {}_xA_z$, we define

$${}_x\widetilde{R}^x_z:H\otimes {}_xA_z\to {}_xA_z\otimes H,\quad {}_x\widetilde{R}^x_z(h\otimes f)=\sum h_{(1)}\cdot f\otimes h_{(2)}.$$

For $x\neq y$ we take ${}_x\widetilde{R}^y_z=0$. It is easy to see that R is a simple twisting system of $\mathbb{K}$-linear categories. Clearly $\mathbb{K}$-$\boldsymbol{Mod}$ is S-distributive, for any set S. If $\mathbb{K}$ is a field then $\mathbb{K}$-$\boldsymbol{Mod}$ is a domain, so in this case the twisted tensor product of $\boldsymbol{A}$ and $\boldsymbol{H}$ with respect to R makes sense, cf.

Theorem 2.14. It is called the smash product of $\boldsymbol{A}$ by H, and it is denoted by $\boldsymbol{A}\#H$. By definition, ${}_x(\boldsymbol{A}\#H)_y = {}_xA_y \otimes H$ and

$$(f \otimes h) \circ (f' \otimes h') = \sum f \circ (h_{(1)} \cdot f') \otimes h_{(2)} h', \tag{37}$$

for any $f \in {}_xA_y$, $f' \in {}_yA_z$ and $h, h' \in H$.

4.5. The semidirect product. Let $\boldsymbol{A}$ be a category. Let us suppose that $(B, \cdot, 1)$ is a monoid that acts to the left on each ${}_xA_y$ via

$$\triangleright : B \times {}_xA_y \to {}_xA_y.$$

We define the category $\boldsymbol{B}$ so that ${}_xB_x = B$ and ${}_xB_y = \emptyset$, for $x \neq y$. The composition of morphisms is given by the multiplication in $\boldsymbol{B}$. To this data we associate a matched pair $(\boldsymbol{A}, \boldsymbol{B}, |\cdots|, \triangleright, \triangleleft)$, setting $f \triangleleft g = f$ for any $(f, g) \in {}_xB_xA_y$, and defining the function $|\cdots| : S^3 \to S$ by $|xyz| = z$. Note that if $x \neq y$ then ${}_xB_yA_z$ is empty, so ${}_x\triangleright_z^y$ and ${}_x\triangleleft_z^y$ coincide with the empty function. One shows that $(\boldsymbol{A}, \boldsymbol{B}, \triangleright, \triangleleft, |\cdots|)$ is a matched pair if and only if for any $(g, g') \in {}_xA_yA_z$ and $f \in B$

$$\begin{aligned} f \triangleright (g \circ g') &= (f \triangleright g) \circ (f \triangleright g'), \\ f \triangleright 1_x^A &= 1_x^A. \end{aligned}$$

The corresponding bicrossed product will be denoted in this case by $\boldsymbol{A} \rtimes B$. If $|S| = 1$ then $\boldsymbol{A}$ can be identified with a monoid and $\boldsymbol{A} \rtimes B$ is the usual semidirect product of two monoids. For this reason we shall call $\boldsymbol{A} \rtimes B$ the semidirect product of $\boldsymbol{A}$ with B. Note that ${}_x(\boldsymbol{A} \rtimes B)_y = {}_xA_y \times B$. For any $f, f' \in B$ and $(g, g') \in {}_xA_yA_z$, the composition of morphisms in $\boldsymbol{A} \rtimes B$ is given by

$$(g, f) \circ (g', f') = (g \circ (f \triangleright g'), f \circ f').$$

4.6. Twisting systems between algebras in $\boldsymbol{M}$. We now consider a twisting system R between two $\boldsymbol{M}$-categories $\boldsymbol{B}$ and $\boldsymbol{A}$ with the property that $S = \{x_0\}$. Obviously, $\boldsymbol{M}$ is S-distributive. We shall use the notation $A = {}_{x_0}A_{x_0}$ and $B = {}_{x_0}B_{x_0}$. The composition map $a := {}_{x_0}a_{x_0}^{x_0}$ and $1_A := 1_{x_0}^A$ define an algebra structure on A. A similar notation will be used for the algebra corresponding to the $\boldsymbol{M}$-category $\boldsymbol{B}$. Let R be a morphism from $B \otimes A$ to $A \otimes B$.

Since ${}_{x_0}\sigma_{x_0}^{x_0}$ is the identity map of $B \otimes A$, by Proposition 2.5, we deduce that ${}_{x_0}R_{x_0}^{x_0} = R$ defines a twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$ if and only if R satisfies the relations (13)-(16) with respect to the unique map $|\cdots| : S^3 \to S$. In turn, they are equivalent to the following identities

$$R \circ (b \otimes A) = (A \otimes b) \circ (R \otimes B) \circ (B \otimes R), \tag{38}$$

$$R \circ (B \otimes a) = (a \otimes B) \circ (A \otimes R) \circ (R \otimes A), \tag{39}$$

$$R \circ (1_B \otimes A) = A \otimes 1_B, \tag{40}$$

$$R \circ (B \otimes 1_A) = 1_A \otimes B. \tag{41}$$

In conclusion, in the case when $|S| = 1$, to give a twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$ is equivalent to give a *twisting map* between the algebras B and A, that is a morphism R which satisfies (38)-(41).

By applying Corollary 2.15 to a twisting map $R : B \otimes A \to A \otimes B$ (viewed as a twisting system between two $\boldsymbol{M}$-categories with one object) we get the *twisted tensor algebra* $A \otimes_R B$. The unit of this algebra is $1_A \otimes 1_B$ and its multiplication is given by

$$m = (a \otimes b) \circ (A \otimes R \otimes B).$$

Note that, in view of the foregoing remarks, an algebra C in $\boldsymbol{M}$ factorizes through A and B if and only if it is isomorphic to a twisted tensor algebra $A \otimes_R B$, for a certain twisting map R.

An algebra in the monoidal category $\mathbb{K}$-$\boldsymbol{Mod}$ is by definition an associative and unital $\mathbb{K}$-algebra. Twisted tensor $\mathbb{K}$-algebras were investigated for instance in [Ma1], [Tam], [CSV], [CIMZ], [LPoV] and [JLPvO].

Coalgebras over a field $\mathbb{K}$ are algebras in the monoidal category $(\mathbb{K}\text{-}\boldsymbol{Mod})^o$. Hence a twisting map between two coalgebras $(A, \Delta_A, \varepsilon_A)$ and $(B, \Delta_B, \varepsilon_B)$ is a $\mathbb{K}$-linear map

$$R : A \otimes_{\mathbb{K}} B \to B \otimes_{\mathbb{K}} A$$

which satisfies the equations that are obtained from (38)-(41) by making the substitutions $a := \Delta_A$, $b := \Delta_B$, $1_A := \varepsilon_A$ and $1_B := \varepsilon_B$, and reversing the order of the factors with respect to the composition in $\boldsymbol{M}$. For example (38) should be replaced with

$$(\Delta_B \otimes_{\mathbb{K}} A) \circ R = (B \otimes_{\mathbb{K}} R) \circ (R \otimes_{\mathbb{K}} B) \circ (A \otimes_{\mathbb{K}} \Delta_B).$$

Obviously $A \otimes_R B$ is the $\mathbb{K}$-coalgebra $(A \otimes_{\mathbb{K}} B, \Delta, \varepsilon)$, where $\varepsilon := \varepsilon_A \otimes \varepsilon_B$ and

$$\Delta = (A \otimes R \otimes B) \circ (\Delta_A \otimes_{\mathbb{K}} \Delta_B).$$

An algebra in Λ-$\boldsymbol{Mod}$-Λ is called a Λ-*ring*. Specializing $\boldsymbol{M}$ to Λ-$\boldsymbol{Mod}$-Λ we find the definition of the *twisted tensor* Λ-*ring*. Dually, Λ-corings are algebras in $(\Lambda\text{-}\boldsymbol{Mod}\text{-}\Lambda)^o$. Thus in this particular case we are led to the construction of the *twisted tensor* Λ-*coring*.

By definition a monad on a category $\boldsymbol{A}$ is an algebra in $[\boldsymbol{A}, \boldsymbol{A}]$. If (F, μ_F, ι_F) and (G, μ_G, ι_G) are monads in $\boldsymbol{M}$, then a natural transformation

$$\lambda : G \circ F \to F \circ G$$

satisfies the relations (38)-(41) if and only if λ is a *distributive law*, cf. [Be]. Let $F^2 := F \circ F$. For every distributive law λ we get a monad $(F \circ G, \mu, \iota)$, where $\iota := \iota_F G \circ \iota_G = F\iota_G \circ \iota_F$ and

$$\mu := \mu_F G \circ F^2 \mu_G \circ F\lambda G = F\mu_G \circ \mu_F G^2 \circ F\lambda G.$$

Distributive laws between comonads can be defined similarly, or working in $[\boldsymbol{A}, \boldsymbol{A}]^o$.

Finally, twisting maps in $\boldsymbol{Opmon(M)}$ have been considered in [BV]. In loc. cit. the authors define a bimonad in $\boldsymbol{M}$ as an algebra in $\boldsymbol{Opmon(M)}$. Hence a twisting map between two bimonads is an opmonoidal distributive law between the underlying monads. For any opmonoidal distributive law λ between the bimonads G and F, there is a canonical bimonad structure on the endofunctor $F \circ G$. See [BV, Section 4] for details.

4.7. Matched pairs of algebras in $\boldsymbol{Coalg(M')}$. Let $\boldsymbol{M}$ denote the category of coalgebras in a braided monoidal category $(\boldsymbol{M'}, \otimes, \mathbf{1}, \chi)$. By definition, a bialgebra in $\boldsymbol{M'}$ is an algebra in $\boldsymbol{M}$. We fix two bialgebras $(A, a, 1_A, \Delta_A, \varepsilon_A)$ and $(A, b, 1_B, \Delta_B, \varepsilon_B)$ in $\boldsymbol{M'}$ and take $R : B \otimes A \to A \otimes B$ to be a morphism in $\boldsymbol{M}$. By Lemma 3.1, there are the coalgebra maps $\triangleright : B \otimes A \to A$ and $\triangleleft : B \otimes A \to B$ such that

$$R = (\triangleright \otimes \triangleleft) \circ \Delta_{B \otimes A} \qquad \text{and} \qquad \chi_{A,B} \circ (\triangleright \otimes \triangleleft) \circ \Delta_{B \otimes A} = (\triangleleft \otimes \triangleright) \circ \Delta_{B \otimes A}. \tag{42}$$

We have seen that R is a twisting map in $\boldsymbol{M}$ if and only if it satisfies the relations (38)-(41). In view of Lemma 3.4, these equations are equivalent to the fact that $(A, \triangleright)$ is a left B-module and $(B, \triangleleft)$ is a right A-module such that the following identities hold:

$$\triangleleft \circ (b \otimes A) = b \circ (\triangleleft \otimes B) \circ (B \otimes \triangleright \otimes \triangleleft) \circ (B \otimes \Delta_{B \otimes A}), \tag{43}$$

$$\triangleright \circ (B \otimes a) = a \circ (A \otimes \triangleright) \circ (\triangleright \otimes \triangleleft \otimes A) \circ (\Delta_{B \otimes A} \otimes A), \tag{44}$$

$$\triangleleft \circ (1_B \otimes A) = \varepsilon_A \otimes 1_B, \tag{45}$$

$$\triangleright \circ (B \otimes 1_A) = 1_A \otimes \varepsilon_B. \tag{46}$$

By definition, a *matched pair of bialgebras in* $\boldsymbol{M'}$ consists of a left B-action $(A, \triangleright)$ and a right A-action $(B, \triangleleft)$ in $\boldsymbol{M}$ such that the second equation in (42) and the relations (43)-(46) hold. For a matched pair of bialgebras we shall use the notation $(A, B, \triangleright, \triangleleft)$.

Summarizing, there is an one to-one-correspondence between twisting maps of bialgebras in $\boldsymbol{M'}$ and matched pairs of bialgebras in $\boldsymbol{M'}$. If $(A, B, \triangleright, \triangleleft)$ is a matched pair of bialgebras and R is the corresponding twisting map, then $A \otimes_R B$ will be called the *bicrossed product of the bialgebras* A and B, and it will be denoted by $A \bowtie B$. Note that $A \bowtie B$ is an algebra in $\boldsymbol{M}$. Thus the bicrossed product of A and B is a bialgebra in $\boldsymbol{M'}$. The unit of this bialgebra is $1_A \otimes 1_B$ and the multiplication is given by

$$m = (a \otimes b) \circ (A \otimes \triangleright \otimes \triangleleft \otimes B) \circ (A \otimes \Delta_{B \otimes A} \otimes B).$$

As a coalgebra $A \bowtie B$ is the tensor product coalgebra of A and B. We also conclude that a bialgebra C in $\boldsymbol{M'}$ factorizes through the sub-bialgebras A and B if and only if $C \cong A \bowtie B$.

As a first application, let us take $\boldsymbol{M'}$ to be the category of sets, which is braided with respect to the braiding given by $(X,Y)\mapsto(Y,X)$ and $(f,g)\mapsto(g,f)$, for any sets X,Y and any functions f,g. We have already noticed that there is a unique coalgebra structure on a given set X

$$\Delta(x)=(x,x),\qquad \varepsilon(x)=\emptyset,$$

where $\emptyset$ denotes the empty set; recall that the unit object in $\boldsymbol{Set}$ is $\{\emptyset\}$. Hence an ordinary monoid, i.e. an algebra in $\boldsymbol{Set}$, has a natural bialgebra structure in this braided category. Moreover any twisting map $R:B\times A\to A\times B$ between two monoids $(A,\cdot,1_A)$ and $(B,\cdot,1_B)$ is a twisting map of bialgebras in $\boldsymbol{Set}$. Let $(A,B,\triangleright,\triangleleft)$ be the corresponding matched pair. One easily shows that the second condition in (42) is always true. By notation, the functions $\triangleright$ and $\triangleleft$ map $(f,g)\in B\times A$ to $f\triangleright g$ and $f\triangleleft g$, respectively. Hence the equations (43)-(46) are equivalent to the following ones:

$$\begin{aligned}(f\cdot f')\triangleleft g&=[f\triangleleft(f'\triangleright g)]\cdot(f'\triangleleft g),\\ f\triangleright(g\cdot g')&=(f\triangleright g)\cdot[(f\triangleleft g)\triangleright g'],\\ 1_B\triangleleft g=1_B\qquad&\text{and}\qquad f\triangleright 1_A=1_A.\end{aligned}$$

Since $R(f,g)=(f\triangleright g,f\triangleleft g)$ the product of the monoid $A\bowtie B$ is defined by the formula

$$(g,f)\cdot(g',f')=\left(g\,(f\triangleright g')\,,(f\triangleleft g')\,f'\right).$$

In conclusion, a monoid C factorizes through A and B if and only if $C\cong A\bowtie B$.

We have seen that the bicrossed product of two groupoids is a groupoid. Thus, if A and B are groups, then $A\bowtie B$ is a group as well. This result was proved by Takeuchi who introduced the matched pairs of groups in [Tak].

We now consider the braided category $\mathbb{K}$-$\boldsymbol{Mod}$, whose braiding is the usual flip map. An algebra in $\boldsymbol{M}$, the monoidal category of $\mathbb{K}$-coalgebras, is a bialgebra over the ring $\mathbb{K}$, and conversely. Proceeding as in the previous case, one shows that a twisting map $R:B\otimes_{\mathbb{K}}A\to A\otimes_{\mathbb{K}}B$ of bialgebras is uniquely determined by the coalgebra maps $\triangleright:B\otimes_{\mathbb{K}}A\to A$ and $\triangleleft:B\otimes_{\mathbb{K}}A\to B$ via the formula

$$R(f\otimes g)=\textstyle\sum(f_{(1)}\triangleright g_{(1)})\otimes(f_{(2)}\triangleleft g_{(2)}).$$

Using the Σ-notation, the second equation in (42) is true if and only if

$$\textstyle\sum(f_{(1)}\triangleleft g_{(1)})\otimes(f_{(2)}\triangleright g_{(2)})=\sum(f_{(2)}\triangleleft g_{(2)})\otimes(f_{(1)}\triangleright g_{(1)}),$$

for any $f\in B$ and $g\in A$. On the other hand, the equations (43)-(46) hold if and only if

$$\begin{aligned}(gg')\triangleleft f&=\textstyle\sum[g\triangleleft(g'_{(1)}\triangleright f_{(1)})](g'_{(2)}\triangleleft f_{(2)}),\\ g\triangleright(ff')&=\textstyle\sum(g_{(1)}\triangleright f_{(1)})[(g_{(2)}\triangleleft f_{(2)})\triangleright f'],\\ f\triangleright 1^A&=\varepsilon_B(f)1^A,\\ 1^B\triangleleft g&=\varepsilon_A(g)1^B,\end{aligned}$$

for any $f,f'\in B$ and any $g,g'\in A$. Thus, we rediscover the definition of *matched pairs of bialgebras* and the formula for the multiplication of the *double cross product*, see [Ma2, Theorem 7.2.2]. Namely,

$$(f\otimes g)(f'\otimes g')=\textstyle\sum f(g_{(1)}\triangleright f'_{(1)})\otimes(g_{(2)}\triangleleft f'_{(2)})g'.$$

4.8. Twisting systems between thin categories. Our aim now is to investigate the twisting systems between two thin categories $\boldsymbol{B}$ and $\boldsymbol{A}$. Recall that a category is thin if there is at most one morphism between any couple of objects. Thus, for any x and y in S we have that either ${}_xA_y=\{{}_xg_y\}$ or ${}_xA_y$ is the empty set. Clearly, if ${}_xA_y=\{{}_xg_y\}$ and ${}_yA_z=\{{}_yg_z\}$ then ${}_xg_y\circ{}_yg_z={}_xg_z$. The identity morphism of x is ${}_xg_x$. Similarly, if ${}_xB_y$ is not empty then ${}_xB_y=\{{}_xf_y\}$.

We fix a twisting system R between $\boldsymbol{B}$ and $\boldsymbol{A}$. It is defined by a family of maps

$${}_xR_z^y:{}_xB_y\times{}_yA_z\to\coprod_{u\in S}A_uB_z$$

that render commutative the diagrams in Figure 3. We claim that R is simple. We need a function $|\cdots|:S^3\to S$ such that the image of ${}_xR_z^y$ is included into ${}_xA_{|xyz|}B_z$ for all $(x,y,z)\in S^3$. Let $T\subseteq S^3$ denote the set of all triples such that ${}_xB_yA_z={}_xB_y\times{}_yA_z$ is not empty. Of course, if

(x, y, z) is not in T then ${}_xR^y_z$ is the empty function, so we can take $|xyz|$ to be an arbitrary element in S. For $(x, y, z) \in T$ there exists $|xyz| \in S$ such that

$$ {}_xR^y_z({}_xf_y, {}_yg_z) = ({}_xg_{|xyz|}, {}_{|xyz|}f_z). \tag{47} $$

Hence ${}_xR^y_z$ is a function from ${}_xB_yA_z$ to ${}_xA_{|xyz|}B_z$. Note that ${}_xA_{|xyz|}B_z$ is not empty in this case. For any $(x, y, z) \in S^3$ we set ${}_x\widetilde{R}^y_z := {}_xR^y_z$. By Proposition 2.5 and Corollary 2.7 it follows that R is simple. We would like now to rewrite the conditions from the definition of simple twisting systems in an equivalent form, that only involves properties of T and $|\cdots|$. For instance let us show that the first condition from Corollary 2.7 is equivalent to:

(i) If $(y, z, t) \in T$ and $(x, y, |yzt|) \in T$ then $|xy|yzt|| = |xzt|$.

Indeed, if ${}_xB_yB_zA_t$ is not empty then ${}_yB_zA_t \neq \emptyset$, so $(y, z, t) \in T$. We have already noticed that ${}_yA_{|yzt|}B_t$ is not empty, provided that ${}_yB_zA_t$ is so. Since ${}_xB_y$ and ${}_yA_{|yzt|}$ are not empty it follows that ${}_xB_yA_{|yzt|}$ has the same property, that is $(x, y, |yzt|) \in T$. Therefore, if ${}_xB_yB_zA_t$ is not empty then $(y, z, t) \in T$ and $(x, y, |yzt|) \in T$. It is easy to see that the reversed implication is also true. Thus it remains to prove that the equation (13) holds in the case when ${}_xB_yB_zA_t$ is not empty. But this is obvious, as ${}_xR^z_t \circ {}_xb^y_zA_t$ and $({}_xA_{|xy|yzt||}b^{|yzt|}_t) \circ {}_xR^y_{|yzt|}B_t \circ {}_xB_yR^z_t$ have the same source ${}_xB_yB_zA_t$ and the same target ${}_xA_{|xzt|}B_t = {}_xA_{|xy|xyz||}B_t$. Both sets are singletons, so the above two morphisms must be equal.

Proceeding in a similar way, we can prove that the other three conditions from Corollary 2.7 are respectively equivalent to:

(ii) If $(x, y, z) \in T$ and $(|xyz|, z, t) \in T$ then $||xyz|\, zt| = |xyt|$.
(iii) If $(x, x, y) \in T$ then $|xxy| = y$.
(iv) If $(x, y, y) \in T$ then $|xyy| = x$.

The last condition in the definition of simple twisting systems is equivalent to:

(v) If $(x, y, z) \in T$ then ${}_xA_{|xyz|}B_z$ is not empty.

Hence for a twisting system R the function $|\cdots|$ satisfies the above five conditions. Conversely, let $|\cdots| : S^3 \to S$ denote a function such that the above five conditions hold. Let ${}_xR^y_z$ be the empty function, if (x, y, z) is not in T. Otherwise we define ${}_xR^y_z$ by the formula (47). In view of the foregoing remarks it is not difficult to see that $R := \{{}_xR^y_z\}_{x,y,z\in S}$ is a simple twisting system.

Clearly two functions $|\cdots|$ and $|\cdots|'$ induce the same twisting system if and only if their restriction to T are equal. Summarizing, we have just proved the theorem below.

Theorem 4.9. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be thin categories. Let T denote the set of all triples $(x, y, z) \in S^3$ such that ${}_xB_yA_z$ is not empty. If R is a twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$ then there exists a function $|\cdots| : S^3 \to S$ such that the conditions (i)-(v) from the previous subsection hold, and conversely. Two functions $|\cdots|$ and $|\cdots|'$ induce the same twisting system R if and only if their restriction to T are equal.*

4.10. The twisted tensor product of thin categories. Let R be a twisting system between two thin categories $\boldsymbol{B}$ and $\boldsymbol{A}$. By the preceding theorem, R is simple and there are T and $|\cdots| : S^3 \to S$ such that the conditions (i)-(v) hold. In particular, the twisted tensor product of these categories exists. By definition, we have ${}_x(\boldsymbol{A} \otimes_R \boldsymbol{B})_y = \coprod_{u\in S}({}_xA_u \times {}_uB_z)$. We can identify this set with

$$ {}_xS_y := \{u \in S \mid {}_xA_uB_y \neq \emptyset\}. $$

For $u \in {}_xS_y$ and $v \in {}_yS_z$ we have $(u, y, v) \in T$. Thus ${}_uR^y_v({}_uf_y, {}_yg_v) = ({}_ug_{|uyv|}, {}_{|uyv|}f_v)$, so the composition in $\boldsymbol{A} \otimes_R \boldsymbol{B}$ is given by

$$ ({}_xg_u, {}_uf_y) \circ ({}_yg_v, {}_vf_z) = ({}_xg_u \circ {}_ug_{|uyv|}, {}_{|uyv|}f_v \circ {}_vf_z) = ({}_xg_{|uyv|}, {}_{|uyv|}f_z). $$

Let $\boldsymbol{C}(S, T, |\cdots|)$ be the category whose objects are the elements of S. By definition, the hom-set ${}_x\boldsymbol{C}(S, T, |\cdots|)_y$ is ${}_xS_y$, the identity map of $x \in S$ is x itself and the composition is given by

$$ \circ : {}_xS_y \times {}_yS_z \to {}_xS_z, \quad u \circ v = |uyv|. $$

Therefore, we have just proved that $\boldsymbol{A} \otimes_R \boldsymbol{B}$ and $\boldsymbol{C}(S, T, |\cdots|)$ are isomorphic.

Remark 4.11. Let $\boldsymbol{C}$ be a small category. Let S denote the set of objects in $\boldsymbol{C}$. The category C factorizes through two thin categories if and only if there are $T \subseteq S$ and $|\cdots| : S^3 \to S$ as in the previous subsection such that $\boldsymbol{C}$ is isomorphic to $\boldsymbol{C}(S, T, |\cdots|)$.

4.12. Twisting systems between posets. Any poset is a thin category, so we can apply Theorem 4.9 to characterize a twisting system R between two posets $\boldsymbol{B} := (S, \preceq)$ and $\boldsymbol{A} := (S, \leq)$. In this setting the corresponding set T contains all $(x, y, z) \in S^3$ such that $x \preceq y$ and $y \leq z$. For simplicity, we shall write this condition as $x \preceq y \leq z$. A similar notation will be used for arbitrarily long sequences of elements in S. For instance, $x \leq y \preceq z \preceq t \leq u$ means that $x \leq y$, $y \preceq z$, $z \preceq t$ and $t \leq u$. The function $|\cdots|$ must satisfies the following conditions:

(i) If $x \preceq y \leq z$ then $x \leq |xyz| \preceq z$.
(ii) If $x \preceq y \preceq z \leq t$ then $|xy|yzt|| = |xzt|$.
(iii) If $x \preceq y \leq z \leq t$ then $||xyz|zt| = |xyt|$.
(iv) If $x \leq y$ then $|xxy| = y$.
(v) If $x \preceq y$ then $|xyy| = x$.

In the case when the posets $\leq$ and $\preceq$ are identical, an example of function $|\cdots| : S^3 \to S$ that satisfies the above conditions is given by $|xyz| = z$, if $y \neq z$, and $|xyz| = x$, otherwise.

4.13. Example of twisting map between two groupoids. Let $\boldsymbol{A}$ be a groupoid with two objects, $S = \{1, 2\}$. The hom-sets of $\boldsymbol{A}$ are the following:

$$ {}_1A_2 = \{u\}, \quad {}_2A_1 = \{u^{-1}\}, \quad {}_1A_1 = \{Id_1\}, \quad {}_2A_2 = \{Id_2\}. $$

Note that $\boldsymbol{A}$ is thin. We set $\boldsymbol{B} := \boldsymbol{A}$ and we take R to be a twisting system between $\boldsymbol{B}$ and $\boldsymbol{A}$. By Theorem 4.9 there are T and $|\cdots| : S^3 \to S$ that satisfies the conditions (i)-(v) in §4.8. Since all sets ${}_xB_yA_z = {}_xB_y \times {}_yA_z$ are nonempty it follows that $T = S$. Thus $|xxy| = y$ and $|xyy| = x$, for all $x, y \in S$. There are two triples $(x, y, z) \in S^3$ such that $x \neq y$ and $y \neq z$, namely $(1, 2, 1)$ and $(2, 1, 2)$. Hence we have to compute $|121|$ and $|212|$. If we assume that $|121| = 1$, then

$$ 1 = |221| = |21|121|| = |211| = 2, $$

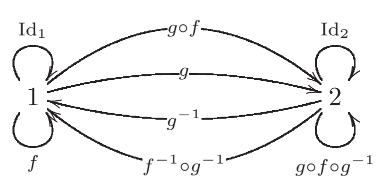
so we get a contradiction. Thus $|121| = 2$, and proceeding in a similar way one proves that $|212| = 1$. It is easy to check that $|\cdots|$ satisfies the required conditions, so there is only one twisting map R between $\boldsymbol{A}$ and itself. Since $\boldsymbol{A}$ is a groupoid, the corresponding bicrossed product $\boldsymbol{C} := \boldsymbol{A} \bowtie \boldsymbol{A}$ is a groupoid as well, see the subsection (4.3). By definition,

$$ {}_1C_1 = \coprod_{x \in \{1,2\}} {}_1A_x \times {}_xA_1 = \{(Id_1, Id_1), (u, u^{-1})\}. $$

Analogously one shows that

$$ {}_1C_2 = \{(Id_1, u), (u, Id_2)\}, \quad {}_2C_1 = \{(Id_2, u^{-1}), (u^{-1}, Id_1)\} \quad \text{and} \quad {}_2C_2 = \{(Id_2, Id_2), (u^{-1}, u)\}. $$

By construction of the twisting map R we get ${}_1R_1^2(u, u^{-1}) \in {}_1A_{|121|} \times {}_{|121|}A_1 = \{(u, u^{-1})\}$. The other maps ${}_xR_z^y$ can be determined analogously. The complete structure of this groupoid is given in the picture below, where we used the notation $f := (u, u^{-1})$ and $g := (Id_1, u)$.



Note that $f^2 = Id_1$ and $g^{-1} = (Id_2, u^{-1})$. Now we can say easily which arrow corresponds to a given morphism in $\boldsymbol{C}$, as in each home-set we have identified at least one element.

Acknowledgments. The first named author was financially supported by the funds of the Contract POSDRU/6/1.5/S/12. The second named author was financially supported by CNCSIS, Contract 560/2009 (CNCSIS code ID_69).

References

[Be] J. Beck, *Distributive laws*, Lecture Notes in Mathematics **80** (1969), 119-140.

[BD] Y. Bespalov and B. Drabant, *Cross product bialgebras*, J. Algebra **240** (2001), No.2, 445-504.

[BV] A. Bruguieres and A. Virelizier, *The double of a Hopf monad*, `arXiv:0812.244.`

[CIMZ] S. Caenepeel, B. Ion, G. Militaru and S. Zhu, *Smash biproducts of algebras and coalgebras, Algebras and Representation Theory* **3** (2000), 19-42.

[CSV] A. Cap, H. Schichl and J. Vanžura, *On twisted tensor products of algebras*, Comm. Algebra **23** (1995), 4701-4735.

[LPoV] J. López Peña, F. Panaite and F. Van Oystaeyen, *General twisting of algebras*, Adv. Math. **212**, No. 1, 315-337 (2007).

[JLPvO] P. Jara Martínez, J. López Peña, F. Panaite and F. Van Oystaeyen, *On iterated twisted tensor products of algebras*, Int. J. Math. **19**, No. 9, 1053-1101 (2008).

[Ke] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, London Mathematical Society Lecture Note Series No. **64**, 1982.

[La] S. Lack, *Composing PROPs*, Theory and Applications of Categories **13** (2004), 147–163.

[Ma1] S. Majid, *Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossed-product construction,* J. Algebra **130** (1990), 17-64.

[Ma2] S. Majid, *Foundations of quantum group theory,* Cambridge Universiy Press XIX, 1995.

[Mai] E. Maillet, *Sur les groupes échangeables et les groupes décomposables*, Bull. Soc. Math. France **28** (1900), 7–16.

[Mar] F. Marmolejo, *Distributive laws for pseudomonads*, Theory and Applications of Categories, **5** (1999), 91–147.

[Ore] O. Ore, *Structures and group theory*, Duke Math. J. **3** (1937), no. 2, 149–174.

[RW] R. Rosebrugh and R.J. Wood, *Distributive laws and factorization*, J. Pure Appl. Algebra **175** (2002), 327–353.

[Sz] J. Szép, *Über die als Produkt zweier Untergruppen darstellbaren endlichen Gruppen*, Comment. Math. Helv. **22** (1949), 31–33.

[Tak] M. Takeuchi, *Matched pairs of groups and bismash products of Hopf algebras*, Comun. Algebra **9** (1981), 841-882.

[Tam] D. Tambara, *The coendomorphism bialgebra of an algebra*, J. Fac. Sci. Univ. Tokyo **37** (1990), 425-456.

[Za] G. Zappa, *Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro*, Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, 119–125. Edizioni Cremonense, Rome, 1942.

University of Bucharest, Faculty of Mathematics and Informatics, Bucharest, 14 Academiei Street, Ro-010014, Romania.

E-mail address: `aura_bardes@yahoo.com`

University of Bucharest, Faculty of Mathematics and Informatics, Bucharest, 14 Academiei Street, Ro-010014, Romania.

E-mail address: `drgstf@gmail.com`